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Dynamics of a Quadratic Map in Two Complex Variables

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DEDICATED TO PROFESSOR JACK K. HALE ON THE OCCASION OF HIS 70TH BIRTHDAY

We study the map Ψ: $\mathbb{C}^2 \to \mathbb{C}^2$ defined by $\Psi(w, z) = (z, z + w^2)$ and the associated

positive sequences, periodic and asymptotically periodic sequences and establish the existence of doubly infinite homoclinic sequences, non-zero sequences whose limits as $j \to \pm\infty$ are 0. We investigate some associated functional equations, $f(x+2) = f(x+1) + f(x)^2$ and $L(L(x)) = L(x) + x^2$.

1. INTRODUCTION

Consider the map Ψ: $\mathbb{C}^2 \to \mathbb{C}^2$ defined by $\Psi(w, z) = (z, z + w^2)$. The map Ψ is one of the simplest possible examples of a nonlinear polynomial map of $\mathbb{C}^2$ into itself which is not a diffeomorphism and, in fact, not even a local homeomorphism near $(0, 0)$. We are interested in understanding the dynamical system determined by iterates $\Psi^{[n]}(w, z)$, $n \geqslant 0$. Even though Ψ is not one-one, much of our analysis will be devoted to the study of certain "natural" biinfinite sequences $\{\zeta_n\}_{n \in \mathbb{Z}}$ such that $\Psi(\zeta_n) = \zeta_{n+1}$ for all $n \in \mathbb{Z}$. Equivalently we shall sometimes study "natural" biinfinite sequences $\{z_n\}_{n \in \mathbb{Z}}$ or infinite sequences $\{z_n\}_{n \geqslant 0}$ such that $z_{n+2} = z_{n+1} + z_n^2$ for all n. We shall refer to these as solutions of the quadratic Fibonacci recurrence or the QF recurrence. Some aspects of this recurrence have already been studied in Duke *et al.* [6] with $\{z_0, z_1\} = \{0, 1\}$, where a combinatorial interpretation of z_n for $n \geqslant 0$ was given and the asymptotic growth of z_n was analyzed.

This is a long paper, so an outline may be in order. We begin by considering Ψ as a map of $\mathbb{R}^2 \to \mathbb{R}^2$. If $(x_0, x_1) \in Q = \{(x, y): x \geqslant 0, y \geqslant 0, (x, y) \neq (0, 0)\}$ in $\mathbb{R}^2$, and $\{x_k\}_{k \geqslant 0}$ is a corresponding solution of the QF recurrence, we

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prove (Corollary 2.1) there there are positive constants $\gamma_e(x_0, x_1) > 1$ and $\gamma_0(x_0, x_1) > 1$ so that

$$\lim_{n\to\infty} \frac{x_{2n}}{\gamma_e(x_0, x_1)^{\sqrt{2}^{2n}}} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{x_{2n+1}}{\gamma_o(x_0, x_1)^{\sqrt{2}^{2n+1}}} = 1.$$

We prove that the maps $(x, y) \to \gamma_e(x, y)$ and $(x, y) \to \gamma_o(x, y)$ extend to an open neighborhood of Q in $\mathbb{C}^2$ and are holomorphic in the neighborhood. Furthermore (Proposition 5.1), $\partial\gamma_e(x, y)/\partial x > 0$, $\partial\gamma_e(x, y)/\partial y \geqslant 0$, $\partial\gamma_o(x, y)/\partial x \geqslant 0$, and $\partial\gamma_o(x, y)/\partial y \geqslant 0$ for all $x, y > 0$. In general, $\gamma_e(x_0, x_1) \neq \gamma_o(x_0, x_1)$.

Section 3 commences the study of a central theme of this paper, doubly infinite sequences which solve the QF recurrence. If $x \geqslant 0$, we prove (see Theorem 3.1 and Theorem 3.2) that there is a *unique* biinfinite sequence of *reals* $\{x_k\}_{k\in\mathbb{Z}}$ with $x_0 = x$ which satisfies the QF recurrence. Furthermore, $\{x_k\}_{k\in\mathbb{Z}}$ is necessarily a sequence of non-negative reals and $\lim_{k\to-\infty} x_k = 0$. With this result we define maps $h_{-1}\colon [0, \infty) \to [0, \infty)$ and $h_1\colon [0, \infty) \to [0, \infty)$ by $h_{-1}(x) = x_{-1}$ and $h_1(x) = x_1$, and sometimes write $L = h_1$. We prove that h_1 and h_{-1} are strictly increasing, continuous maps of $[0, \infty)$ onto $[0, \infty)$, that $h_1(h_{-1}(x)) = h_{-1}(h_1(x)) = x$ for all $x \geqslant 0$, and that, in fact, h_1 and h_{-1} are continuously differentiable: see Theorems 4.1 and 4.2. The argument in Theorem 4.2 that h_1 and h_{-1} are C^1 on $[0, \infty)$ involves continued fractions and expresses $h_1'(x)$ as a function of the map $h_1|_{[0,x]}$ and the formula for h_1' is a nonstandard functional differential equation. The continued fraction approach has drawbacks, however, and in Section 10 we use a different method to prove that h_1 and h_{-1} are C^∞ on $[0, \infty)$. Lemma 3.6, Proposition 4.1, and results in Section 10 lead to methods for the approximation of L. In particular, we show that $R(x) = \log(\gamma_e(x, L(x)))/\log(\gamma_o(x, L(x)))$ is always equal to 1 to six decimal place accuracy, but is *not* identically equal to 1.

Let $\mathscr{H}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $\mathscr{H}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$, $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$, $\mathbb{R}_{\leqslant 0} = \{x \in \mathbb{R} : x \leqslant 0\}$, and $\mathbb{C}_- = \mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$. For $z \in \mathbb{C}\backslash\{0\}$ let $\arg z \in (-\pi, \pi]$ be the unique $\theta = \arg z$ with $z = |z| \exp(i\theta)$. We prove (see Theorem 8.1) that there is an injective holomorphic map $h_{-1}\colon \mathbb{C}_- \to \mathbb{C}_-$ such that $h_{-1}(\mathscr{H}_+) \subset \mathscr{H}_+$, $h_{-1}(\mathbb{R}_{>0}) \subset \mathbb{R}_{>0}$, $h_{-1}(\bar{z}) = \overline{h_{-1}(z)}$ for all $z \in \mathbb{C}_-$ and h_{-1} agrees with the previously defined h_{-1} on $\mathbb{R}_{>0}$. If $h_j(z) = h_{-1}^{[|j|]}(z)$ for $j < 0$ (the composition of h_{-1} with itself $|j|$ times), $h_0(z) = z$, and $h_j(z) = h_{j-1}(z) + (h_{j-2}(z))^2$ for $j > 0$, then h_j is holomorphic on $\mathbb{C}_-$ for all $j \in \mathbb{Z}$, $h_j(\mathbb{C}_-) \subset \mathbb{C}_-$ for $j < 0$, h_1 agrees with the previously defined h_1 on $\mathbb{R}_{>0}$, and $h_1(h_{-1}(z)) = z$ for all $z \in \mathbb{C}_-$.

If $z \in \mathscr{H}_+$ and we define $\{z_j\}_{j\in\mathbb{Z}}$ by $z_j = h_j(z)$, then $\{z_j\}$ is a solution of the QF recurrence, $z_0 = z$, $z_j \in \mathscr{H}_+$ for $j \leqslant 0$, $0 < \arg z_{j-1} < \arg z_j <$

$2 \arg z_{j-1}$ for $j \leqslant 0$, $\lim_{j\to-\infty} \arg z_j = 0$ and $\lim_{j\to-\infty} z_j = 0$: see Proposition 8.1 and Theorem 8.1. A biinfinite sequence which satisfies the QF recurrence and possesses the preceding properties is called an "argument increasing recurrence sequence through z" or an "*AIR* sequence through z": see Definition 8.2 for the precise definition.

The previous results imply that h_1 and h_{-1} are real analytic on $\mathbb{R}_{>0}$. It is easy to see that $L = h_1$ satisfies the equation $x^2 + L(x) = L(L(x))$ for $x \geqslant 0$, and using this equation one can write a formal Taylor series for L at 0. Nevertheless, we prove (Theorem 9.2) that neither h_1 nor h_{-1} can be extended to be holomorphic on an open neighborhood of 0, even though (see Lemma 9.3) h_1 and h_{-1} are bounded on bounded subsets of $\mathbb{C}_-$.

In Theorem 11.1, we prove that there is an open set U in $\mathbb{C}^2$ which contains $(0, 0)$ in its closure and satisfies $\Psi(U) \subset (U)$ and $\lim_{k\to\infty} \Psi^{[k]}(w, z) = (0, 0)$ for all $(w, z) \in U$. Combining this result with facts about *AIR* sequences we prove the existence of special biinfinite sequences homoclinic to 0: if $\frac{3\pi}{4} < \theta < \pi$, it is proved in Theorem 11.2 that there exists $\rho(\theta) > 0$ such that if $z = re^{i\theta}$ and $0 < r < \rho(\theta)$, then there is an *AIR* sequence $\{z_j\}_{j\in\mathbb{Z}}$ through z with $0 < \arg z_j < \arg z_{j+1} < \pi$ for all $j \in \mathbb{Z}$, $\lim_{j\to-\infty} z_j = 0$, and $\lim_{j\to\infty} z_j = 0$.

By using the real analyticity of γ_e, γ_o, and h_1, we prove in Section 4 that, for each $t > 0$, there is a real analytic, positive, strictly increasing map $f\colon \mathbb{R} \to \mathbb{R}_{>0}$ with $f(x+2) = f(x+1) + f(x)^2$ for all $x \in \mathbb{R}$ (a continuous version of the basic recurrence relationship) and $f(0) = t$. One can easily see that $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. We are also able to describe other solutions to this functional equation.

The above summary emphasizes the maps γ_e, γ_o, and L and *AIR* sequences. There are, however, interesting biinfinite sequences $\{z_j\}_{j\in\mathbb{Z}}$ which satisfy the QF recurrence and are not *AIR* sequences. In Theorem 7.1, we prove that for any $(w, z) \in \mathbb{C}^2$, $(w, z) \neq (0, 0)$, there exists a biinfinite sequence $\{z_j\}_{j\in\mathbb{Z}}$ with $z_0 = w$ and $z_1 = z$ which satisfies the QF recurrence and is such that either $z_{2n} \to 1 + i$ and $z_{2n+1} \to 1 - i$ as $n \to -\infty$ or $z_{2n} \to 1 - i$ and $z_{2n+1} \to 1 + i$ as $n \to -\infty$.

We discuss periodic points of Ψ in Section 6. If $\Psi^{[n]}$ denotes the nth iterate of Ψ, we use topological degree to prove that $\Psi^{[n]}(\zeta) = \zeta$ has, counting algebraic multiplicity, 2^n solutions in $\mathbb{C}^2$. We prove that for each prime p, Ψ has a periodic point of minimal period p, so Ψ has infinitely many distinct periodic points. We also give a precise analysis of all fixed points of $\Psi^{[n]}$ for $1 \leqslant n \leqslant 4$.

There is an extensive literature concerning the dynamics determined by polynomial diffeomorphisms of $\mathbb{C}^n$ (see the early paper [9], and, more recently, [2, 3, and 7] and the references there). Many of the results in this literature depend on assumptions of hyperbolicity and use ergodic theory. Since our map Ψ is not a diffeomorphism and is not hyperbolic at its fixed point (0, 0), much of this literature is not directly relevant. Other recent work is relevant. The map $\Psi^{[2]}$ is a nondegenerate polynomial endomorphism of $\mathbb{C}^2$ in the sense of Peng [18] and also is regular following the definition of Bedford and Jonsson [1]. These authors and others study maps like $\Psi^{[2]}$ by extending them to complex projective space and using detailed analyses of Green functions. Some of our results overlap with those obtained by these authors, but our methods applied to Ψ allow us to provide more specific conclusions. Additionally, the types of questions we ask and the types of results we obtain are different.

As already noted, the map Ψ is not a local diffeomorphism at its fixed point (0, 0) and the Fréchet derivative of Ψ at (0, 0) has eigenvalues 0 and 1. In this situation, a general result of O. E. Lanford yields a local C^k "center manifold" but provides no global results and no analyticity results. Our map $h_{-1}: \mathbb{C}_- \to \mathbb{C}_-$ gives a different kind of global center manifold with analyticity properties. If we define Σ_j for $j = -1, -2$ by $\Sigma_j = \{(h_j(z), h_{j+1}(z)): z \in \mathbb{C}_-\}$, then $\Sigma_2 \subset \Sigma_1$, $\Psi(\Sigma_2) = \Sigma_1$ and Σ_1 can be considered a subset of a center manifold for Ψ at (0, 0). It seems likely that $\bigcup_{n=0}^{\infty} \Psi^{[n]}(\Sigma_2)$, which is invariant under Ψ, is quite complicated. The ideas which we use can be applied to other examples where standard theory is not applicable, and we hope to pursue this point in a future paper.

This paper is dedicated to Jack Hale on the occasion of his seventieth birthday and in recognition of his many mathematical achievements.

2. THE REAL CASE

Suppose $H_{n+2} = H_{n+1} + H_n^2$ for $n \geqslant 0$ where $H_0, H_1 \geqslant 0$.

LEMMA 2.1. *Suppose there is* $n > 0$ *and a real number* $c_n > 0$ *so that* $H_n \leqslant c_n H_{n-1}^2$ *and* $H_{n+1} \leqslant c_n H_n^2$. *Then* $H_{n+2} \leqslant (c_n^2/(c_n+1))\, H_{n+1}^2$ *and* $H_{n+3} \leqslant (c_n^2/(c_n+1))\, H_{n+2}^2$.

Proof. We need just show that $H_{n+2} \leqslant (c_n^2/(c_n+1))\, H_{n+1}^2$, because then $H_{n+2} \leqslant (c_n^2/(c_n+1))\, H_{n+1}^2 \leqslant c_n H_{n+1}^2$ and $H_{n+1} \leqslant c_n H_n^2$, so the same argument implies that $H_{n+3} \leqslant (c_n^2/(c_n+1))\, H_{n+2}^2$.

By assumption we have

$$H_{n+2} = H_{n+1} + H_n^2 \leqslant (c_n + 1)\, H_n^2$$

and

$$H_{n+1}^2 = (H_n + H_{n-1}^2)^2 \geqslant \left(H_n + \frac{1}{c_n} H_n\right)^2 = \frac{(c_n+1)^2}{c_n^2} H_n^2.$$

Thus $H_{n+2} \leqslant \kappa H_{n+1}^2$ if $\kappa((c_n+1)^2/c_n^2)\, H_n^2 \geqslant (c_n+1)\, H_n^2$ which is true when $\kappa = c_n^2/(c_n+1)$. ▮

The following result follows from Lemma 2.1.

LEMMA 2.2. *Suppose there is* $n>0$ *and a real number* $c_n>0$ *so that* $H_n \leqslant c_n H_{n-1}^2$ *and* $H_{n+1} \leqslant c_n H_n^2$. *Then* $H_{n+2m} \leqslant c_n^{2^m} H_{n+2m-1}^2$ *and* $H_{n+2m+1} \leqslant c_n^{2^m} H_{n+2m}^2$ *for any positive integer* m.

Proof. Lemma 2.1 verifies this claim for $m=1$ since $c_n^2/(c_n+1) \leqslant c_n^2$. The general case can be proved by induction. ▮

If we know $H_n \leqslant c_n H_{n-1}^2$ and $H_{n+1} \leqslant c_n H_n^2$, with $c_n \geqslant 1$, then we can eventually get a similar estimate for some larger n with positive constant less than 1 using the following consequence of Lemma 2.1.

LEMMA 2.3. *Suppose there is* $n>0$ *and a real number* $c_n>0$ *so that* $H_n \leqslant c_n H_{n-1}^2$ *and* $H_{n+1} \leqslant c_n H_n^2$, *then for any positive integers* m,

$$H_{n+2m} \leqslant \left(\frac{c_n}{c_n+1}\right)^m c_n H_{n+2m-1}^2 \quad \text{and} \quad H_{n+2m+1} \leqslant \left(\frac{c_n}{c_n+1}\right)^m c_n H_{n+2m}^2.$$

Proof. Lemma 2.1 proves these statements when $m=1$. If we now assume the result for a positive integer m, and let $\tilde{c} = (c_n/(c_n+1))^m\, c_n$, then Lemma 2.1 shows that $H_{n+2m+2} \leqslant (\tilde{c}/(\tilde{c}+1))\, \tilde{c}\, H_{n+2m+1}^2 \leqslant (c_n/(c_n+1))(c_n/(c_n+1))^m\, c_n H_{n+2m+1}^2 = (c_n/(c_n+1))^{m+1}\, c_n H_{n+2m+1}^2$ as desired because $x/(x+1)$ is increasing and $\tilde{c} < c_n$. ▮

Therefore if $H_n \leqslant c_n H_{n-1}^2$ and $H_{n+1} \leqslant c_n H_n^2$ with $c_n \geqslant 1$, there is some sufficiently large m with $H_{n+2m} \leqslant \frac{1}{2} H_{n+2m-1}^2$ and $H_{n+2m+1} \leqslant \frac{1}{2} H_{n+2m}^2$. To see this, take any m with $(c_n/(c_n+1))^m\, c_n \leqslant \frac{1}{2}$. Such m exists since $0 < c_n/(c_n+1) < 1$.

Suppose we select $N>0$ so that $H_N \leqslant c_N H_{N-1}^2$ and $H_{N+1} \leqslant c_N H_N^2$ *with* $0 < c_N < 1$. The previous estimates then verify these inequalities:

$$\begin{aligned}
H_{N+2} &= H_{N+1} + H_N^2 \leqslant (1+c_N)\, H_N^2 \\
H_{N+4} &= H_{N+3} + H_{N+2}^2 \leqslant (1+c_N^2)\, H_{N+2}^2 \leqslant (1+c_N^2)(1+c_N)^2 H_N^{2^2} \\
H_{N+6} &= H_{N+5} + H_{N+4}^2 \leqslant (1+c_N^{2^2})\, H_{N+4}^2 \\
&\leqslant (1+c_N^{2^2})(1+c_N^2)^2\, (1+c_N)^{2^2}\, H_N^{2^3}.
\end{aligned}$$

Generally, if we know

$$H_{N+2m} \leqslant (1+c_N^{2^{m-1}})(1+c_N^{2^{m-2}})^2\, (1+c_N^{2^{m-3}})^{2^2} \cdots (1+c_N)^{2^{m-1}}\, H_N^{2^m}$$

then since $H_{N+2m+2} \leqslant (1+c_N^{2^m})\, H_{N+2m}^2$ we see

$$H_{N+2m+2} \leqslant (1+c_N^{2^m})(1+c_N^{2^{m-1}})^2\,(1+c_N^{2^{m-2}})^{2^2}\,(1+c_N^{2^{m-3}})^{2^3} \cdots (1+c_N)^{2^m}\, H_N^{2^{(m+1)}}$$

so by induction this holds for all $m \geqslant 0$.

Since $c_N^j \leqslant c_N$ for all $j \geqslant 0$ because we assumed $c_N < 1$, we can further simplify our estimate:

$$H_{N+2m} \leqslant (1+c_N)(1+c_N)^2\,(1+c_N)^{2^2} \cdots (1+c_N)^{2^{m-1}}\, H_N^{2^m} \leqslant (1+c_N)^{2^m}\, H_N^{2^m}.$$

But $H_{n+2} \geqslant H_n^2$ always, so we can estimate from below very simply: $H_N^{2^m} \leqslant H_{N+2m}$ for any $m > 0$.

If we define γ_N by $\gamma_N = H_N^{2^{-N/2}}$, then for $m \geqslant 0$ we know

$$\gamma_N^{2^{(N+2m)/2}} \leqslant H_{N+2m} \leqslant (\gamma_N (1+c_N)^{2^{-N/2}})^{2^{(N+2m)/2}}.$$

The following result summarizes and extends this discussion and also asserts that the *even* and *odd* growth constants of $\{H_n\}$ which are called below γ_e and γ_o are well-defined.

THEOREM 2.1. *Suppose $H_0, H_1 \geqslant 0$ and $H_{n+2} = H_{n+1} + H_n^2$ for $n \geqslant 0$. Given $\varepsilon > 0$, there exists $N \geqslant 1$ and a constant c_N with $0 < c_N < \varepsilon$ so that*

$$H_N \leqslant c_N H_{N-1}^2 \qquad \textit{and} \qquad H_{N+1} \leqslant c_N H_N^2. \qquad (\#)$$

Also, if $m \geqslant 0$,

$$H_{N+2m} \leqslant c_N^{(2^m)} H_{N+2m-1}^2 \qquad \textit{and} \qquad H_{N+2m+1} \leqslant c_N^{(2^m)} H_{N+2m}^2. \qquad (\#\#)$$

Furthermore, if $\gamma_N = H_N^{2^{-N/2}}$, then

$$\gamma_N^{(2^{(N+2m)/2})} \leqslant H_{N+2m} \leqslant (\gamma_N (1+c_N)^{2^{-N/2}})^{(2^{(N+2m)/2})}.$$

Also, $\{\gamma_{2n}\}$ is increasing with limit γ_e and $\{\gamma_{2n+1}\}$ is increasing with limit γ_o.

Proof. The previous discussion proves (#) and (##) and verifies that the c_N's can be chosen so that $c_N \to 0$ as $n \to \infty$. Indeed, the lemmas show that the c_N's can be chosen to approach 0 very rapidly. Then (##) states that

$$\gamma_N \leqslant \gamma_{N+2m} \leqslant \gamma_N (1+c_N)^{2^{-N/2}}$$

for $m \geqslant 0$. This shows that the sequences $\{\gamma_{2n}\}$ and $\{\gamma_{2n+1}\}$ are both Cauchy and bounded above. The sequences are increasing because

$$\gamma_{n+2}^{(2^{(n+2)/2})} = H_{n+2} = H_{n+1} + H_n^2 \geqslant H_n^2 = (\gamma_n^{(2^{n/2})})^2 = \gamma_n^{(2^{(n+2)/2})}$$

so that $\gamma_{n+2} \geqslant \gamma_n$. The even and odd subsequences of $\{\gamma_n\}$ are each increasing and bounded, so the existence of γ_e and γ_o is guaranteed. ∎

The *growth constants* γ_e and γ_o of the sequence $\{H_n\}$ depend on the initial conditions H_0 and H_1, and this dependence may be indicated by $\gamma_e = \gamma_e(H_0, H_1)$ and $\gamma_o = \gamma_o(H_0, H_1)$. These growth constants are positive when H_0 and H_1 are both non-negative and not both 0.

COROLLARY 2.1. *If H_0 and H_1 are non-negative and not both* 0, *then* $\lim_{n\to\infty} H_{2n}/(\gamma_e(H_0, H_1)^{\sqrt{2}^{2n}}) = 1$ *and* $\lim_{n\to\infty} H_{2n+1}/(\gamma_o(H_0, H_1)^{\sqrt{2}^{2n+1}}) = 1$.

Proof. We prove the result for $\gamma_e(H_0, H_1)$. The argument for $\gamma_o(H_0, H_1)$ is similar. (# #) implies that

$$\gamma_{2n} \leqslant \gamma_e(H_0, H_1) \leqslant \gamma_{2n}(1 + c_{2n})^{2^{-n}},$$

where $\gamma_{2n}^{2^n} = H_{2n}$. Therefore

$$\frac{1}{1 + c_{2n}} \leqslant \frac{\gamma_{2n}^{2^n}}{(\gamma_e(H_0, H_1))^{2^n}} = \frac{H_{2n}}{(\gamma_e(H_0, H_1))^{2^n}} \leqslant 1.$$

We have already observed that $\lim_{n\to\infty} c_{2n} = 0$ so the even part of the corollary is proved. ∎

Theorem 2.1 can be used to get accurate information about the asymptotic growth of $\{H_n\}$. In particular, the estimates reveal distinct analytic growth rates of the even and odd subsequences.

EXAMPLE 2.1. If $H_0 = 0$ and $H_1 = 1$, then $H_{11} = 207\ 73703$ and $H_{12} = 1\ 15957\ 36272$ and $H_{13} = 43155\ 83320\ 68481$ so that $H_{12} \leqslant c_{12} H_{11}^2$ and $H_{13} \leqslant c_{12} H_{12}^2$ with $c_{12} = \frac{1}{3} \cdot 10^{-4}$. Theorem 2.1 then implies that for $n \geqslant 6$

$$\gamma_{12}^{(2^n)} \leqslant H_{2n} \leqslant (\gamma_{12}(1 + \tfrac{1}{3} \cdot 10^{-4})^{(2^{-12/2})})^{(2^n)},$$

where $(\gamma_{12})^{64} = H_{12}$.

Since $H_{14} = 1\ 34461\ 53124\ 81085\ 26465$ we can see that $H_{13} \leqslant c_{13} H_{12}^2$ and $H_{14} \leqslant c_{13} H_{13}^2$ hold with $c_{13} = 4 \cdot 10^{-6}$. Therefore Theorem 2.1 states that for $n \geqslant 6$

$$\gamma_{13}^{(2^{(2n+1)/2})} \leqslant H_{2n+1} \leqslant (\gamma_{13}(1 + 4 \cdot 10^{-6})^{(2^{-13/2})})^{(2^{(2n+1)/2})}$$

with $(\gamma_{13})^{13/2} = H_{13}$. Further direct computation shows that $\gamma_{12} = 1.43633\ 14218\ 20338$ and $\gamma_{13} = 1.45109\ 50811\ 54281$, each with error less than 10^{-15}. Therefore

$$(1.43633\ \ 14218)^{\sqrt{2}^{2n}} \leqslant H_{2n} \leqslant (1.43633\ \ 21699)^{\sqrt{2}^{2n}}$$

and

$$(1.45109\ \ 50811)^{\sqrt{2}^{2n+1}} \leqslant H_{2n+1} \leqslant (1.45109\ \ 51453)^{\sqrt{2}^{2n+1}}$$

are true for all $n \geqslant 6$, and thus

$$\gamma_e(0, 1) \in [1.43633\ \ 14218, 1.43633\ \ 21699] \qquad \text{and}$$

$$\gamma_o(0, 1) \in [1.45109\ \ 50811, 1.45109\ \ 51453].$$

These intervals are disjoint, so $\gamma_o(0, 1) \neq \gamma_e(0, 1)$.

We can extend these results in several ways. First, it is not at all obvious that there are solutions to our recurrence which are always *negative*. Two iterations of the recurrence relation transform the initial conditions (H_0, H_1) to the pair $(H_0^2 + H_1, H_0^2 + H_1^2 + H_1)$. So after H_0 (for $n > 0$) the sequence must be nondecreasing. Viewing H_3 as a function of H_0 and H_1 shows that *any* initial conditions in the plane outside a circle of radius $\frac{1}{2}$ centered at $(0, -\frac{1}{2})$ give rise to a sequence which has positive terms for $n \geqslant 2$. Carefully chosen negative initial conditions produce sequences which are always negative.

THEOREM 2.2. *Suppose that* $p > 1$ *and* q *is any real number satisfying* $0 < q \leqslant p^{-1} - p^{-2}$. *If* $H_0 = -q$ *and* $H_1 = -qp^{-1}$ *and* $H_{n+2} = H_{n+1} + H_n^2$ *for* $n \geqslant 0$ *then for all* $n \geqslant 0$, $H_n \leqslant p^{-1} H_{n-1}$ *and* $H_n \leqslant -qp^{-n}$.

Proof. Suppose we know the following inductive hypotheses,

$$H_j \leqslant p^{-1} H_{j-1} \qquad \text{and} \qquad H_j \leqslant -qp^{-j},$$

for $0 < j \leqslant k$. These assumptions are certainly valid for $k = 1$. Then $H_k \leqslant p^{-1} H_{k-1} < 0$ so $(pH_k)^2 \geqslant H_{k-1}^2$ and thus

$$H_{k+1} = H_k + H_{k-1}^2 \leqslant H_k(1 + p^2 H_k).$$

Here $H_0 \leqslant H_1 \leqslant \cdots \leqslant H_k < 0$ using an inductive hypothesis and the recurrence equation. Then $p^2 H_1 = -pq \leqslant p^2 H_k$. Note that $0 < pq < 1$ since q is selected so that $pq \leqslant 1 - p$ so that $0 < 1 - pq = 1 + p^2 H_1 \leqslant 1 + p^2 H_k$

because H_k is negative, larger than H_1 and has absolute value smaller than 1. Thus the estimate above proves that $H_{k+1} \leqslant (1 - pq)\, H_k$ again because H_k is negative.

Finally, $0 < p^{-1} \leqslant 1 - pq$ so that $H_{k+1} \leqslant p^{-1} H_k \leqslant -qp^{k+1}$, completing a proof by induction of the theorem. ∎

Here is a geometric translation of this theorem and its preliminary remarks. Suppose that $H_0 = -q \to x$ and $H_1 = -qp^{-1} \to y$ and we graph the allowed initial conditions (x, y) which result in a sequence $\{H_n\}$ with all H_n's negative. The restriction $p > 1$ with x and y negative restricts (x, y) to lie in the third quadrant above the main diagonal. The more complicated restriction $q \leqslant p^{-1} - p^{-2}$ becomes $-x^3 \leqslant xy - y^2$ in these coordinates. The shaded area in Fig. 1 is a sketch of the region, $\mathscr{R}$, defined by this inequality. $\mathscr{R}$ must lie inside the circle mentioned above (the left half of that circle is drawn) and above the diagonal line drawn. The boundary of $\mathscr{R}$ has a vertical tangent when $x = -\frac{1}{4}$ at the point $(-\frac{1}{4}, -\frac{1}{8})$. The theorem asserts that if H_0 is any number between 0 and $-(\max_{p>1} p^{-1} - p^{-2}) = -\frac{1}{4}$ then there are numbers between H_0 and 0 so that the recurrence has negative solutions for all $n \geqslant 0$.

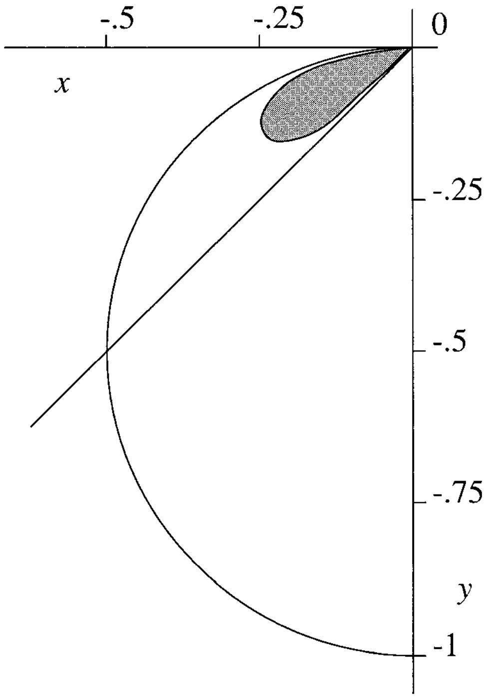

FIG. 1. The region $\mathscr{R}$.

Other initial conditions not in $\mathscr{R}$ may determine sequences satisfying our recurrence with all terms negative. For example, $(-0.3, -0.2)$ is not in $\mathscr{R}$, but one iteration of the map $(x, y) \to (y, y+x^2)$ produces $(-0.2, -0.11)$, which is in the region. Not every point in the lune between the line and the circle has an iterate in $\mathscr{R}$: $(-0.48, -0.46)$ is in the lune, but $H_3 > 0$ for these initial conditions. We have not analyzed sequences which change sign. Other aspects of the iteration are studied in what follows, but certainly the dynamics of the mapping are not completely understood.

3. DOUBLY INFINITE SEQUENCES

We now describe doubly infinite real sequences $\mathbf{X}$ which satisfy the QF recurrence. These $\mathbf{X}$'s are sequences $\{x_n\}_{n \in \mathbb{Z}}$ for which $x_{n+2} = x_{n+1} + x_n^2$ for all $n \in \mathbb{Z}$. The existence of non-zero doubly infinite sequences which satisfy the recurrence is not obvious. The arguments presented here are elementary but a bit intricate. We begin by considering those pairs (x_0, x_1) which have infinitely many ancestors.

LEMMA 3.1. *Suppose that a real sequence* $\{x_n\}_{n \leqslant 1}$ *satisfies the recurrence* $x_{n+2} = x_{n+1} + x_n^2$ *for all* $n \leqslant -1$. *Then* $\{x_n\}$ *is increasing*: $x_n \leqslant x_{n+1}$ *for all* $n \leqslant 0$. *Also*, $x_n \geqslant 0$ *for all* $n \leqslant 1$ *and* $\lim_{n \to -\infty} x_n = 0$. *If any* $x_n = 0$ *then all* $x_n = 0$.

Proof. Such a sequence must be increasing since $x_n^2 \geqslant 0$ always. Since $\{x_n\}$ is increasing, $\lim_{n \to -\infty} x_n = \inf_{n \leqslant 1} x_n = L$. Either $L = -\infty$ or L is finite.

We first show that L cannot be $-\infty$. Select $M \geqslant \frac{1}{2}$ so that $M^2 - M > x_1$. Since $\xi \to \xi^2 - \xi$ is increasing on $[\frac{1}{2}, \infty)$, $\xi^2 - \xi > x_1$ for all $\xi \geqslant M$. If $L = -\infty$, we can select $N \leqslant 1$ so that $x_N < -M$. Since $x_{N-1} \leqslant x_N < 0$, $x_{N-1}^2 \geqslant x_N^2$. If we then take $\xi = -x_N$, $x_{N+1} = x_N + x_{N-1}^2 \geqslant -\xi + \xi^2 > x_1$, contradicting the increasing nature of $\{x_n\}$. So L must be finite.

We know that $L = \lim_{n \to -\infty} x_{n+2} = \lim_{n \to -\infty} x_{n+1} + x_n^2 = L + L^2$ so that L must be 0. Therefore $x_n \geqslant 0$ for all n. If there is N so that $x_N = 0$, then $x_n = 0$ for all $n \leqslant N$ because the sequence is increasing and bounded below by 0. The descendents $x_1, x_2, ..., x_{N-1}$ must all be 0 by using the recurrence with initial data $(x_{N+1}, x_N) = (0, 0)$. ∎

If $\mathbf{X}$ is a doubly infinite sequence satisfying our recurrence, we will need to describe x_n as a function of x_0 and x_1 for n positive and n negative. Since $x_{n+2} = x_{n+1} + x_n^2$ we know that $x_n = \sqrt{x_{n+2} - x_{n+1}}$. We may take the square root to be non-negative because of the preceding lemma. We can go backwards when $x_{n+2} \geqslant x_{n+1}$, which occurs when $\mathbf{X}$ satisfies our recurrence. For $n \geqslant 2$, x_n is a continuous function of x_0 and x_1, and we write $x_n = F_n(x_0, x_1)$. F_n is a polynomial. Let $K^2 = \{(x_0, x_1) \in \mathbb{R}^2 : x_0 \geqslant 0, x_1 \geqslant 0\}$. If $x = (x_0, x_1) \in K^2$ and $y = (y_0, y_1) \in K^2$, we write $x \leqslant y$ if $x_0 \leqslant y_0$ and $x_1 \leqslant y_1$. F_n is a strictly increasing function using this partial order: if $x, y \in K^2$ with $x \leqslant y$ and $x \neq y$, then $F_n(x) < F_n(y)$ for all $n \geqslant 2$.

LEMMA 3.2. *Suppose $\delta > 0$ and $N \geqslant 2$ is a positive integer. Let $\Gamma_N(\delta) = \{(x_0, x_1) \in K^2 : x_0 \leqslant x_1 \text{ and } F_N(x_0, x_1) = \delta\}$. Then $\Gamma_N = \Gamma_N(\delta)$ is a compact, connected, nonempty set, and $I_N(\delta) = \{F_{N+1}(x_0, x_1) : (x_0, x_1) \in \Gamma_N\}$ is a compact, nonempty interval.*

Proof. Define J_δ to be $\{(x_0, x_1) \in K^2 : 0 \leqslant x_0 \leqslant x_1 \leqslant \delta\}$. If $(x_0, x_1) \in K^2$ and $x_1 > \delta$ then $F_N(x_0, x_1) > \delta$. Thus $\Gamma_N \subset J_\delta$. Since F_N is continuous and J_δ is compact, Γ_N is also compact. For each m in the closed unit interval we consider the line segment $\ell_m = \{(x_0, x_1) \in J_\delta : x_0 = m x_1\}$. If we restrict F_N to ℓ_m and note that $F_N(0, 0) = 0$ and $F_N(m\delta, \delta) > \delta$, the Intermediate Value Theorem implies that there is a positive number $\rho_N(m)$ so that $F_N(\rho_N(m)(m\delta, \delta)) = \delta$. Since we know that the function $\rho \to F_N(\rho(m\delta, \delta))$ defined for $0 \leqslant \rho \leqslant 1$ is strictly increasing, the number $\rho_N(m)$ is unique.

If the mapping $m \to \rho_N(m)$ is not continuous for $m \in [0, 1]$, there is a sequence $\{m_j\}$ in $[0, 1]$ so that $m_j \to m$ but $\{\rho_N(m_j)\}$ does not converge to $\rho_N(m)$. We may take a subsequence and assume that $\rho_N(m_j) \to \tilde{\rho} \neq \rho_N(m)$. Since F_N is continuous, we know that $\delta = \lim_{j \to \infty} F_N(\rho_N(m_j)(m_j\delta, \delta)) = F_N(\tilde{\rho}(m\delta, \delta))$. But $\delta = F_N(\rho_N(m)(m\delta, \delta))$ also. Since $\delta = F_N(\rho(m\delta, \delta))$ must have a unique solution, we have deduced a contradiction.

The line segments ℓ_m sweep out all of J_δ, and therefore $\Gamma_N = \{\rho(m)(m\delta, \delta) : 0 \leqslant m \leqslant 1\}$. Since $m \to \rho(m)(m\delta, \delta) \in J_\delta$ is continuous, the set Γ_N is the continuous image of $[0, 1]$ and is therefore compact, connected, and nonempty.

The set $I_N(\delta)$ is the image of Γ_N under the continuous mapping F_{N+1}, and so must be a compact, connected, nonempty subset of $\mathbb{R}$: a compact, nonempty interval. ∎

We label the endpoints of the interval just found: $I_N(\delta) = [L_N(\delta), R_N(\delta)]$, so $L_N(\delta) \leqslant R_N(\delta)$. We next investigate the dependence of these functions on δ and N.

LEMMA 3.3. *If $0 < \delta_1 < \delta_2$ and N is a positive integer, then $L_N(\delta_1) \leqslant L_N(\delta_2)$ and $R_N(\delta_1) \leqslant R_N(\delta_2)$.*

Proof. We prove the result indicated for L_N. The proof for R_N is similar. Suppose $0 < \delta_1 < \delta_2$, and select Δ larger than both δ_1 and δ_2. The proof of the previous lemma implies that given $m \in [0, 1]$, there are unique numbers $\rho_1(m)$ and $\rho_2(m)$ so that $F_N(\rho_1(m)(m\Delta, \Delta)) = \delta_1$ and $F_N(\rho_2(m)(m\Delta, \Delta)) = \delta_2$. Since $F_N(\rho(m\Delta, \Delta))$ is an increasing function of ρ, we know that $\rho_1(m) < \rho_2(m)$.

Now $L_N(\delta_j) = \min\{F_{N+1}(x_0, x_1)\colon (x_0, x_1) \in \Gamma_N(\delta_j)\}$. Since each $\Gamma_N(\delta_j)$ is compact, there must be $(y_j, z_j) \in \Gamma_N(\delta_j)$ where each minimum is attained: $L_N(\delta_j) = F_{N+1}(y_j, z_j)$. By our remarks in the previous paragraph, there is $s \in (0, 1)$ so that $s(y_2, z_2) \in \Gamma_N(\delta_1)$. Therefore $L_N(\delta_2) = F_{N+1}(y_2, z_2) \geqslant F_{N+1}(s(y_2, z_2)) \geqslant \min\{F_{N+1}(x_0, x_1)\colon (x_0, x_1) \in \Gamma_N(\delta_1)\} = L_N(\delta_1)$. ▮

We further characterize the sets $I_N(\delta)$ in terms of ancestors. It will then be easy to observe that these sets are nested: $I_N(\delta) \supset I_{N+1}(\delta)$ for $N \geqslant 2$.

LEMMA 3.4. *Fix $\delta > 0$ and an integer $N \geqslant 1$. Then* $I_N(\delta) =$

$$\{\sigma \in \mathbb{R} : \delta \leqslant \sigma \text{ and there exist } \{y_m\}_{-N \leqslant m \leqslant 1}$$
$$\text{so that } 0 \leqslant y_{-N} \leqslant y_{-N+1} \leqslant \cdots \leqslant y_0 \leqslant y_1$$
$$\text{with } (y_0, y_1) = (\delta, \sigma) \text{ and } y_{m+2} = y_{m+1} + y_m^2 \text{ for } -N \leqslant m \leqslant -1\}$$

Proof. Let W be the set defined by the right-hand side of the preceding equation. If $\sigma \in W$, then define $x_m = y_{m-N}$ for integer m satisfying $0 \leqslant m \leqslant N+1$. Then $F_N(x_0, x_1) = \delta$ and $(x_0, x_1) \in K^2$, so that (x_0, x_1) must be in $\Gamma_N(\delta)$. Therefore $F_{N+1}(x_0, x_1) = \sigma$ must be in $I_N(\delta)$. Thus $W \subset I_N(\delta)$.

On the other hand, if $(x_0, x_1) \in \Gamma_N(\delta)$ and $x_{m+2} = x_{m+1} + x_m^2$ when $0 \leqslant m \leqslant N-1$, we may define y_{m-N} by requiring that $y_{m-N} = x_m$ when $0 \leqslant m \leqslant N+1$. Since $y_0 = \delta$, all the conditions for showing that $y_1 = F_{N+1}(y_{-N}, y_{-N+1}) \in W$ are verified, so that $I_N(\delta) \subset W$. ▮

Now let $\mathscr{F} = \{\mathbf{X} = \{x_n\}_{n \in \mathbb{Z}} : x_{n+2} = x_{n+1} + x_n^2$ for all $n\}$ and let $\mathscr{F}_\delta$ be those elements $\mathbf{X}$ of $\mathscr{F}$ with $x_0 = \delta$. If $\mathscr{F}_\delta$ is not empty, Lemma 3.1 shows that δ must be non-negative, and that the sequence $\mathbf{X}$ is increasing with $\lim_{n \to -\infty} x_n = 0$. It also asserts that the only $\mathbf{X} \in \mathscr{F}$ with any element equal to 0 is the sequence all of whose elements are 0. Of course, the values of x_0 and x_1 determine all elements of any $\mathbf{X} \in \mathscr{F}$.

THEOREM 3.1. *Suppose $\delta > 0$. Then $\mathscr{F}_\delta$ is nonempty, and there is a compact nonempty interval $I_\infty(\delta) = [L(\delta), R(\delta)] = [L, R]$ so that if $\mathbf{X} \in \mathscr{F}_\delta$ then $x_1 \in [L, R]$ and, furthermore, given any $\sigma \in [L, R]$ there is $\mathbf{X} \in \mathscr{F}_\delta$ with $x_1 = \sigma$.*

Proof. Since $I_N(\delta) \supset I_{N+1}(\delta)$ and each $I_N(\delta)$ is a nonempty compact interval, $\bigcap_{N \geqslant 2} I_N(\delta)$ is then a compact nonempty interval. We call this interval $I_\infty(\delta) = [L(\delta), R(\delta)]$.

If $\mathbf{X} \in \mathscr{F}_\delta$, then for every positive integer N, the pair $(x_0, x_1) = (\delta, \sigma)$ must have ancestors of order N, and so σ must be in $I_N(\delta)$ for all N, and therefore $\sigma \in I_\infty(\delta)$.

Given $\sigma \in I_\infty(\delta)$, then for any positive integer N, $\sigma \in I_N(\delta)$ so that there are numbers $y_{-N} \leqslant y_{-N+1} \leqslant \cdots \leqslant y_0 = \delta \leqslant y_1 = \sigma$ with $y_{n+2} = y_{n+1} + y_n^2$ for $-N \leqslant n \leqslant 0$. The numbers $\{y_{-N}, y_{-N+1}, ..., y_{-1}\}$ are *uniquely determined* by δ and σ. Therefore an $\mathbf{X} \in \mathscr{F}_\delta$ with $(x_0, x_1) = (\delta, \sigma)$ can be defined in the following way: create x_n for $n > 1$ by running the recurrence $x_{n+2} = x_{n+1} + x_n^2$ forward with initial conditions (δ, σ). For $n < 0$, create x_n by choosing any $N > -n$ and obtaining the unique numbers $\{y_{-N}, y_{-N+1}, ..., y_{-1}\}$ described above. Take x_n to be y_n. Any choice of N gives the same value for x_n. Since the sequence of y_n's always satisfies our recurrence, we know that the doubly infinite sequence $\mathbf{X}$ is an element of $\mathscr{F}_\delta$ as desired. ∎

We will show that $L = \text{Left}$ and $R = \text{Right}$ as defined above are equal. That is, each $\mathscr{F}_\delta$ has one element for any $\delta \geqslant 0$. This result follows from the oscillation lemma below, which also will help us to approximate the common value of $L(\delta)$ and $R(\delta)$ and, generally, to compare solutions of the QF recurrence.

Lemma 3.5. *Suppose* $\{x_{-N}, x_{-N-1}, ..., x_1\}$ *and* $\{y_{-N}, y_{-N-1}, ..., y_1\}$ *are non-negative solutions of the QF recurrence for some positive integer N. Suppose additionally that* $x_0 \leqslant y_0$ *and* $x_1 > y_1$. *If N is odd then* $x_{-N} > y_{-N}$. *If N is even then* $x_{-N} < y_{-N}$.

Proof. $x_{-1} = \sqrt{x_1 - x_0}$ and $y_{-1} = \sqrt{y_1 - y_0}$, so $x_{-1} > y_{-1}$. Then $x_{-2} = \sqrt{x_0 - x_{-1}}$ and $y_{-2} = \sqrt{y_0 - y_{-1}}$, so that $x_{-2} < y_{-2}$. We now proceed by induction. One of two cases is done here (the other is obtained by interchanging x and y). Assume that $x_{-k-1} < y_{-k-1}$ and $y_{-k-2} > x_{-k-2}$. Then $\sqrt{x_{-k-1} - x_{-k-2}} < \sqrt{y_{-k-1} - y_{-k-2}}$ so $x_{-k-3} < y_{-k-3}$. ∎

Let us consider backwards solutions to the QF recurrence given initial conditions (y_0, y_1) with $y_0 = \delta > 0$ and $y_1 > \delta$, but with $y_1 \notin I_\infty(\delta)$. Then we know there must be positive solutions $\mathbf{Y} = \{y_{-N}, y_{-N+1}, ..., y_0, y_1\}$ for some positive integer N which are *maximal* or else $\mathbf{Y}$ can be extended backwards forever (since we can always propagate $\mathbf{Y}$ forwards and $\mathbf{Y} \notin \mathscr{F}_\delta$). But y_{-N-1} "should be" $\sqrt{y_{-N+1} - y_{-N}}$. So the obstacle must be $y_{-N+1} \leqslant y_{-N}$. If $\mathbf{X} \in \mathscr{F}_\delta$ then either $x_1 > y_1$ or $x_1 < y_1$. Consider the first alternative. The preceding lemma asserts that if N is odd, then $y_{-N+1} > x_{-N+1}$ and $x_{-N} > y_{-N}$. Since $\mathbf{X} \in \mathscr{F}_\delta$, we can *always* propagate backwards, so $x_{-N+1} > x_{-N}$. Combining these inequalities gives the contradiction: $y_{-N+1} > y_{-N}$. So N must be even. Half of the following lemma is now verified, and the proof of the other part is similar.

LEMMA 3.6. *Suppose that* $\mathbf{Y} = \{y_{-N}, y_{-N+1}, ..., y_0, y_1\}$ *is a positive solution to the QF recurrence for some positive integer* N *with* $y_0 = \delta$ *and that* $y_{-N+1} \leqslant y_{-N}$, *so necessarily* $y_1 \notin I_\infty(\delta)$. *If* N *is odd, then* $y_1 > R(\delta)$. *If* N *is even, then* $y_1 < L(\delta)$.

EXAMPLE 3.1. Take $\delta = 1$. The initial conditions $(y_0, y_1) = (1, 2)$ lead to $y_{-1} = 1$ and $y_0 = y_{-1}$. Since 1 is odd, $R(1) < 2$. The initial conditions $(y_0, y_1) = (1, \frac{5}{4})$ lead to $y_{-1} = \frac{1}{2}$ and $y_{-2} = \frac{1}{\sqrt{2}}$, so $y_{-2} > y_{-1}$. Therefore $\frac{5}{4} < L(1)$, and $I_\infty(1) \subset [\frac{5}{4}, 2]$. Further numerical work shows that $I_\infty(1) \subset [1.507, 1.508]$, which certainly suggests the following result.

THEOREM 3.2. *If* $\delta \geqslant 0$, *then* $L(\delta) = R(\delta)$, *so that* $I_\infty(\delta)$ *is one point and* $\mathscr{F}_\delta$ *contains exactly one sequence.*

Proof. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are unequal elements of $\mathscr{F}_\delta$ with, say, $x_1 > y_1$. Then $x_{-n} > y_{-n}$ for all positive odd integers n and $x_{-n} < y_{-n}$ for all positive even integers n. Also recall that both sequences have limit 0 as $n \to -\infty$. Now we compare the differences between the sequences.

$$\begin{aligned} |x_{-n-1} - y_{-n-1}| &= |\sqrt{x_{-n+1} - x_{-n}} - \sqrt{y_{-n+1} - y_{-n}}| \\ &= \frac{|(x_{-n+1} - x_{-n}) - (y_{-n+1} - y_{-n})|}{\sqrt{x_{-n+1} - x_{-n}} + \sqrt{y_{-n+1} - y_{-n}}} \\ &= \frac{|(x_{-n+1} - y_{-n+1}) - (x_{-n} - y_{-n})|}{\sqrt{x_{-n+1} - x_{-n}} + \sqrt{y_{-n+1} - y_{-n}}} \end{aligned}$$

Abridge this by writing $D_k = |x_k - y_k|$ and noting that the signs of $x_k - y_k$ must differ for any consecutive integers. The equation above leads to

$$\begin{aligned} D_{-n-1} &= \frac{D_{-n+1} + D_{-n}}{\sqrt{x_{-n+1} - x_{-n}} + \sqrt{y_{-n+1} - y_{-n}}} \\ &\geqslant \frac{D_{-n}}{\sqrt{x_{-n+1} - x_{-n}} + \sqrt{y_{-n+1} - y_{-n}}} \end{aligned}$$

which is impossible. To see this, consider any two positive sequences, $\{s_n\}$ and $\{t_n\}$, which have limit 0. The inequality $s_{n+1} \geqslant s_n / t_n$ cannot be valid for all sufficiently large n: there must be N so that $t_n < \frac{1}{2}$ when $n \geqslant N$, and then $s_n \geqslant 2^{n-N} s_N$ for all $n \geqslant N$, contradicting the convergence of $\{s_n\}$. ∎

The unique sequence in $\mathscr{F}_\delta$ will be called $\mathbf{X}_\delta$. Further information about the function $L(\delta)$ and an application to functional equations are given in the next section.

4. SOME SOLUTIONS OF AN ASSOCIATED FUNCTIONAL EQUATION

We consider the functional equation

$$f(x+2)=f(x+1)+f(x)^2 \tag{$*$}$$

whose natural initial conditions are functions on $[0, 2)$. We describe some solutions to $(*)$ other than the trivial solution $f\equiv 0$. These solutions use our previous study of the QF recurrence together with further results.

EXAMPLE 4.1. We create a function $f\in C^\infty[0, 2]$ with special behavior at 0 and 2. Suppose $f(0)=0$ and $f(t)>0$ if $t>0$, with $f(1)=f(2)=1$. We also require that f be infinitely flat at 0: all of f's derivatives at 0 are 0. The classical theorem of E. Borel says that all the derivatives of f at 2 may be freely specified, so we may require $f^{(n)}(2)=f^{(n)}(1)$ for all $n\geqslant 0$. Figure 2 is a sketch of one such function.

With these initial conditions, f is a solution of the functional equation $(*)$ which is positive and C^∞ on the half-line $[a, \infty)$ for any $a>0$. Theorem 2.2 proves the existence of negative sequences solving the QF recurrence, and the discussion after the theorem provides a region $\mathscr{R}$ of initial conditions corresponding to those sequences. Any interval of initial conditions smoothly chosen from the region $\mathscr{R}$ can be used in a fashion similar to the preceding example to produce solutions of $(*)$ which are C^∞ and negative on half-lines.

Other solutions to $(*)$ can be created using the function $L(\delta)$ introduced earlier. If $\delta\geqslant 0$, then $\mathbf{X}_\delta$ is the unique doubly infinite sequence in $\mathscr{F}_\delta$. The

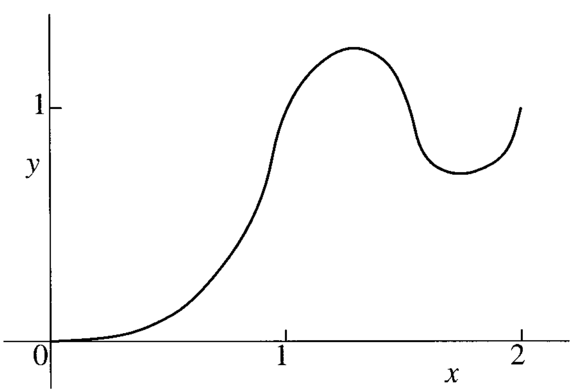

FIG. 2. Non-negative smooth initial conditions.

initial conditions of $\mathbf{X}_\delta$ are $(\delta, L(\delta))$. For $\delta>0$, the sequence has all positive terms and is strictly increasing, so $L(\delta)>\delta$. We will create additional solutions to $(*)$ with domain all of $\mathbb{R}$ using the function L. These solutions will have properties which reflect what is known about L.

THEOREM 4.1. *$L(\delta)$ is strictly increasing and continuous for $\delta \geqslant 0$. $L(0)=0$ and $L(\delta)>\delta$ for $\delta>0$.*

Proof. Recall that $L(\delta)=\lim_{N\to\infty} I_N(\delta)$ so that L is a pointwise limit of increasing functions (Lemma 3.3) and therefore must be increasing. Suppose $0<\delta_1$, $0<\delta_2$, and $L(\delta_1)=L(\delta_2)$. If ρ is this common value, then ρ is positive. Consider $L(\rho)$. There must be exactly one sequence satisfying the QF recurrence with $x_0=\rho$ and $x_1=L(\rho)$. But by uniqueness, x_{-1} must be equal to both δ_1 and δ_2. Therefore L is strictly increasing.

We verify continuity at 0. Suppose some sequence $\{\delta_n\}$ has limit 0 but (passing to a subsequence) $L(\delta_n)\geqslant c>0$. Then consider $x_{\delta_n, -1}$, the -1th entry in the sequence $\mathbf{X}_{\delta_n}$. It must be $\sqrt{L(\delta_n)-\delta_n}$ and less than δ_n. As $n\to\infty$ a contradiction $(c=0)$ appears.

Now suppose $\{\delta_n\}$ is a sequence of non-negative real numbers with $\lim_{n\to\infty}\delta_n=\delta>0$. The sequence $\{L(\delta_n)\}$ is bounded since L is increasing, and so must have convergent subsequences. We are done if we prove that the limit of any subsequence is $L(\delta)$. So, passing to a subsequence, we add the assumption that $\lim_{n\to\infty} L(\delta_n)=\rho$. Consider the -1th term of $\mathbf{X}_{\delta_n}$. Surely $x_{\delta_n, -1}=\sqrt{L(\delta_n)-\delta_n}$. The limit of the right-hand side as $n\to\infty$ is $\sqrt{\rho-\delta}$, and $\rho-\delta\geqslant 0$ since it is the limit of a positive sequence. In fact, a inductive proof shows that for each $m\in\mathbb{Z}$, the sequence $\{x_{\delta_n, m}\}$ must converge to some non-negative number which we call y_m, and that $y_{m+2}=y_{m+1}+y_m^2$. Since $y_0=\delta$, the sequence *must* be the unique element $\mathbf{X}_\delta$ of $\mathscr{F}_\delta$, and therefore $\rho=y_1=L(\delta)$. ∎

The function L satisfies an associated functional equation. We characterize it in several ways as the only non-trivial solution of this equation.

COROLLARY 4.1. *The function L satisfies the equation $\delta^2+L(\delta)=L(L(\delta))$ for all $\delta\geqslant 0$. If $f\colon(0,\infty)\to(0,\infty)$ is a map onto $(0,\infty)$ and if $\delta^2+f(\delta)=f(f(\delta))$ for all $\delta>0$, then $f(\delta)=L(\delta)$ for all $\delta>0$. If $g\colon(0,\infty)\to(0,\infty)$ is continuous and $\delta^2+g(\delta)=g(g(\delta))$ for all $\delta>0$, then $g(\delta)=L(\delta)$ for all $\delta>0$.*

Proof. If $\delta>0$, then $(\delta, L(\delta))$ are the initial conditions for $\mathbf{X}_\delta$. An index shift reveals that $(L(\delta), \delta^2+L(\delta))$ must be the initial conditions for $\mathbf{X}_{L(\delta)}$, but by uniqueness this pair must be the same as $(L(\delta), L(L(\delta)))$.

The functional equation $\delta^2+f(\delta)=f(f(\delta))$ for all $\delta>0$ easily implies that f is one-one, and since we assume that f is onto, we can define the inverse map $f^{[-1]}\colon(0,\infty)\to(0,\infty)$. If $f^{[k]}$ is the k-fold composition of f

with itself for $k>0$ and the k-fold composition of $f^{[-1]}$ with itself for $k<0$, then define, for $\delta>0$, $x_k=f^{[k]}(\delta)$. The doubly infinite sequence $\{x_k\}_{k\in\mathbb{Z}}$ must satisfy $x_k^2+x_{k+1}=x_{k+2}$ for all $k\in\mathbb{Z}$ because of the functional equation for f. Also, $x_0=\delta$ and all of the x_k's are positive. It follows from Theorem 3.2 that such a sequence is unique, so $f(\delta)=L(\delta)$.

If g satisfies the functional equation, g is one-one. Because g is assumed continuous, a familiar calculus lemma implies that either g is strictly increasing on $(0,\infty)$ or g is strictly decreasing on $(0,\infty)$. If g were strictly decreasing, we could write

$$0<\alpha=\lim_{\delta\to 0^+} g(\delta)\leqslant\infty.$$

We may take limits in the functional equation for g as $\delta\to 0^+$ and obtain $\alpha=g(\alpha)$ where this equation also makes sense if $\alpha=\infty$. However, the equation $\alpha=g(\alpha)$ is impossible: if $0<\delta<\alpha$, then $\alpha>g(\delta)>g(\alpha)$. Thus we conclude that g is strictly increasing.

If we write $\alpha=\lim_{\delta\to 0^+} g(\delta)$ again and take limits as $\delta\to 0^+$ in the functional equation for g, we obtain $\alpha=\lim_{\delta\to 0^+} g(g(\delta))$. If $\alpha>0$, we again see that $g(\alpha)=\alpha$. This is impossible, since $\alpha<g(\delta)<g(\alpha)$ for $0<\delta<\alpha$. Thus α must be 0. Since $g(g(\delta))>g(\delta)$ and g is strictly increasing, we see that $g(\delta)>\delta$ always, so $\lim_{\delta\to\infty} g(\delta)=\infty$. Since g is continuous, the Intermediate Value Theorem implies that g maps $(0,\infty)$ onto $(0,\infty)$. The previous result proved here implies that g and L agree on $(0,\infty)$. ∎

If k is a positive integer, define $L^{[k]}(\delta)$ to be $(L\circ L\circ\cdots\circ L)(\delta)$ (k-fold composition). When k is a negative integer, $L^{[k]}$ will be the inverse of L composed with itself k times. The uniqueness used in the preceding proof shows that $\mathbf{X}_\delta=\{L^{[k]}(\delta)\}_{k\in\mathbb{Z}}$. Since both L and its inverse are strictly increasing, if $0\leqslant\delta_1<\delta_2$, then $L^{[k]}(\delta_1)<L^{[k]}(\delta_2)$ for all k.

We now show that L is differentiable. It is helpful to "predict" its derivative. Since $\delta^2+L(\delta)=L(L(\delta))$, if $\rho=L(\delta)$ then $L(\rho)=\rho+(L^{[-1]}(\rho))^2$. If we *assume* L is C^1 with non-zero derivative, then we may differentiate the equation and repeatedly replace L' by the right-hand side of the first equality, appropriately shifted.

$$L'(\rho)=1+\frac{2L^{[-1]}(\rho)}{L'(L^{[-1]}(\rho))}=1+\cfrac{2L^{[-1]}(\rho)}{1+\cfrac{2L^{[-2]}(\rho)}{L'(L^{[-2]}(\rho))}}$$

$$=1+\cfrac{2L^{[-1]}(\rho)}{+\cfrac{2L^{[-2]}(\rho)}{1+\cfrac{2L^{[-3]}(\rho)}{L'(L^{[-3]}(\rho))}}}=\cdots$$

The limit of the expressions indicated above is classically called a *simregular infinite continued fraction.* We establish some notation for such continued fractions.

Given any complex numbers $\alpha_1, \alpha_2, ..., \alpha_n, ...$ define

$$[[\alpha_1]] = 1 + \alpha_1 \qquad \text{and} \qquad [[\alpha_1, \alpha_2, ..., \alpha_n]] = 1 + \frac{\alpha_1}{[[\alpha_2, ..., \alpha_n]]}$$

(if each division is defined) and

$$[[\alpha_1, \alpha_2, ..., \alpha_n, ...]] = \lim_{n \to \infty} [[\alpha_1, \alpha_2, ..., \alpha_n]]$$

when the limit exists.

We rewrite the equation for $L'(\rho)$ using this notation:

$$\begin{aligned} L'(\rho) &= \left[\left[\frac{2L^{[-1]}(\rho)}{L'(L^{[-1]}(\rho))} \right]\right] = \left[\left[2L^{[-1]}(\rho), \frac{2L^{[-2]}(\rho)}{L'(L^{[-2]}(\rho))} \right]\right] \\ &= \left[\left[2L^{[-1]}(\rho), 2L^{[-2]}(\rho), \frac{2L^{[-3]}(\rho)}{L'(L^{[-3]}(\rho))} \right]\right] = \cdots. \end{aligned}$$

A theorem of Worpitzky published in 1865 applies (see, for example, [21], Chapter 2, Section 10). We know that given $R > 0$ and $\rho \in [0, R]$, then $0 \leqslant L^{[-k]}(\rho) \leqslant L^{[-k]}(R)$ and $\lim_{k \to \infty} L^{[-k]}(R) = 0$. Worpitzky's Theorem merely needs $|L^{[-k]}(\rho)| \leqslant \frac{1}{4}$ for sufficiently large k and all ρ under consideration to conclude that the limit displayed below exists, and it then asserts that convergence is uniform in $[0, R]$ to a limit which must be continuous.

$$\begin{aligned} &\lim_{n \to \infty} \left[\left[2L^{[-1]}(\rho), 2L^{[-2]}(\rho), 2L^{[-3]}(\rho), ..., \frac{2L^{[-n]}(\rho)}{L'(L^{[-n]}(\rho))} \right]\right] \\ &\quad = [[2L^{[-1]}(\rho), 2L^{[-2]}(\rho), 2L^{[-3]}(\rho), ..., 2L^{[-n]}(\rho), ...]]. \end{aligned}$$

We still must show that the difference quotient for L has the predicted limit. Therefore we consider $(L(\tilde{\delta}) - L(\delta))/(\tilde{\delta} - \delta)$ for δ and $\tilde{\delta}$ in $[0, R]$ with $\delta \neq \tilde{\delta}$. This quotient is always positive since L is strictly increasing. We manipulate a generalization of this quotient as in the proof of Theorem 3.2.

$$\begin{aligned} L^{[k]}(\tilde{\delta}) - L^{[k]}(\delta) &= \sqrt{L^{[k+2]}(\tilde{\delta}) - L^{[k+1]}(\tilde{\delta})} - \sqrt{L^{[k+2]}(\delta) - L^{[k+1]}(\delta)} \\ &= \frac{(L^{[k+2]}(\tilde{\delta}) - L^{[k+1]}(\tilde{\delta})) - (L^{[k+2]}(\delta) - L^{[k+1]}(\delta))}{\sqrt{L^{[k+2]}(\tilde{\delta}) - L^{[k+1]}(\tilde{\delta})} + \sqrt{L^{[k+2]}(\delta) - L^{[k+1]}(\delta)}} \\ &= \frac{(L^{[k+2]}(\tilde{\delta}) - L^{[k+2]}(\delta)) - (L^{[k+1]}(\tilde{\delta}) - L^{[k+1]}(\delta))}{\sqrt{L^{[k+2]}(\tilde{\delta}) - L^{[k+1]}(\tilde{\delta})} + \sqrt{L^{[k+2]}(\delta) - L^{[k+1]}(\delta)}} \end{aligned}$$

which yields

$$1+\left(\frac{L^{[k]}(\tilde{\delta})-L^{[k]}(\delta)}{L^{[k+1]}(\tilde{\delta})-L^{[k+1]}(\delta)}\right)\left(\sqrt{L^{[k+2]}(\tilde{\delta})-L^{[k+1]}(\tilde{\delta})}\right.$$
$$\left.+\sqrt{L^{[k+2]}(\delta)-L^{[k+1]}(\delta)}\right)$$
$$=\frac{L^{[k+2]}(\tilde{\delta})-L^{[k+2]}(\delta)}{L^{[k+1]}(\tilde{\delta})-L^{[k+1]}(\delta)}.$$

We define $DQ(k)$ to be $(L^{[k]}(\tilde{\delta})-L^{[k]}(\delta))/(L^{[k-1]}(\tilde{\delta})-L^{[k-1]}(\delta))$ so that $DQ(1)$ is the original difference quotient. If $k\in\mathbb{Z}$, $DQ(k)$ is certainly continuous and positive for all $(\tilde{\delta},\delta)\in[0,R]\times[0,R]\setminus\{\tilde{\delta}=\delta\}$. Also, define $M(k)$ to be $\sqrt{L^{[k]}(\tilde{\delta})-L^{[k-1]}(\tilde{\delta})}+\sqrt{L^{[k]}(\delta)-L^{[k-1]}(\delta)}=L^{[k-2]}(\tilde{\delta})+L^{[k-2]}(\delta)$. Then $M(k)$ is continuous and non-negative in all of $[0,R]\times[0,R]$. The previous displayed equation can be rewritten as

$$1+(DQ(k+1))^{-1}\,M(k+2)=DQ(k+2).$$

This equation implies that $DQ(j)>1$ for all $j\in\mathbb{Z}$. By using the equation iteratively we obtain

$$DQ(1)=1+(DQ(0))^{-1}\,M(1)=1+((1+(DQ(-1))^{-1}\,M(0)))^{-1}\,M(1)=\text{etc.}$$

so that if N is any positive integer,

$$DQ(1)=\left[\left[\;M(1),M(0),...,M(-n),...,\frac{M(-N)}{DQ(-N-1)}\right]\right].$$

Notice that as $N\to\infty$, Worpitzky's Theorem implies that the right-hand side converges uniformly to the infinite continued fraction

$$[[M(1),M(0),...,M(-n),...]]$$

which is continuous in all of $[0,R]\times[0,R]$. Therefore the limit as $\tilde{\delta}\to\delta$ of $DQ(1)$ exists. Note also that when $\delta=\tilde{\delta}$ and n is an integer, $M(n)=2L^{[n-2]}(\delta)$. We have verified the following result.

THEOREM 4.2. *L: $[0,\infty)\to[0,\infty)$ is a C^1 bijection, with $L'(\delta)>1$ for all $\delta>0$ and $L'(0)=1$. Also, for all $\delta\geqslant 0$,*

$$L'(\delta)=[[2L^{[-1]}(\rho),2L^{[-2]}(\rho),...,2L^{[-n]}(\rho),...]],$$

where the right-hand side converges uniformly on compact subsets of $[0,\infty)$.

More information about the smoothness of L is obtained in Section 8 when the complex version of this function is discussed. We have been able to verify by direct computation that L is C^2 but the computations are quite tedious.

EXAMPLE 4.2. We specify a solution f of $(*)$ beginning with its behavior on $[0, 1]$. Let f increase continuously from $f(0) = \xi > 0$ to $f(1) = L(\xi) > \xi$ in $[0, 1]$. Define $f(t)$ for $t \in [1, 2]$ to be $L(f(t-1))$. Use this function f as initial data for $(*)$, and use $(*)$ and properties of L to extend f to a positive increasing continuous solution of $(*)$ with domain all of $\mathbb{R}$. The range of f must be all positive real numbers since $f(n) = L^{[n]}(\xi)$ for $n \in \mathbb{Z}$ so that $f(n)$ has limit 0 as $n \to -\infty$ and limit ∞ as $n \to \infty$.

If $f'(1) = L'(\xi) f'(0)$, then f is C^1. All positive, continuous, strictly increasing solutions of $(*)$ are similar to f. If g is any such function, its range must again be all positive numbers. We can then translate g (replacing $g(t)$ by $g(t + t_0)$) so that $g(0) = f(0)$. There must be a homeomorphism $\phi\colon [0, 1] \to [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ so that $g(t) = f(\phi(t))$ for $t \in [0, 1]$, and this easily extends to relate f and g on all of $\mathbb{R}$.

Each solution of $(*)$ in turn yields a curve in $\mathbb{R}^2$: $C_f(t) = (f(t), f(t+1))$. The curve passes through the space of initial conditions of the QF recurrence. If $\Psi(x, y) = (y, y + x^2)$, then f is a solution of $(*)$ if and only if $\Psi \circ C_f(t) = C_f(t+1)$. If f is a positive continuous solution of $(*)$ with domain $\mathbb{R}$, the image of C_f in $\mathbb{R}^2$ is always the graph of L.

L is central to understanding solutions of $(*)$. Lemma 3.6 has already been used in Example 3.1 to approximate $L(1)$. The following result also follows from Lemma 3.6. More refined estimates for L are given in Section 10.

PROPOSITION 4.1. *If* $\frac{1}{2} < c \leqslant 1$, *then* $L(x) \leqslant x + c^2 x^2$ *for all sufficiently large* x. *Also, if* $1 \leqslant d < 2$, *then* $dx \leqslant L(x)$ *for all sufficiently large* x. *The inequalities* $x \leqslant L(x) \leqslant x + x^2$ *are valid for all* $x \geqslant 0$.

Proof. We verify the first assertion since the others follow in a similar fashion. Suppose that $c \geqslant 0$ and x is large and positive. When the QF recurrence is run "backwards" we have $w_{-n-2} = \sqrt{w_{-n} - w_{-n-1}}$. If we begin with initial conditions $w_0 = x$ and $w_1 = x + c^2 x^2$, then $w_{-1} = cx$ and $w_{-2} = \sqrt{(1-c)\,x}$. Certainly if $c \leqslant 1$ the formula for w_{-2} describes an eligible ancestor (in the non-negative reals) for the pair $(x, x + c^2 x^2)$. Also $w_{-3} = \sqrt{cx - \sqrt{(1-c)\,x}}$ which is approximately $\sqrt{cx}$ for c fixed and x large and positive. Finally, $w_{-4} \approx \sqrt{(\sqrt{1-c} - \sqrt{c})\,x}$. Lemma 3.6 asserts that $x + c^2 x^2$ is an overestimate of $L(x)$ if w_{-4} is *not* an eligible ancestor. This occurs when $\sqrt{1-c} - \sqrt{c} < 0$ which happens if $c \in (\frac{1}{2}, 1]$. ∎

Since $L(1) \approx 1.50787\ 47554$, the phrases "sufficiently large" are needed: take $x=1$, $d=1.6$, and $c=0.6$ in the preceding inequalities, for example. Since $L(x) \in [x, x+x^2]$ and Lemma 3.6 allows us to decide if a number is larger or smaller than $L(x)$, a computer program using bisection can approximate $L(x)$ to arbitrary accuracy. In particular, we can create a graph of L. Figure 3 is a graph of L on the interval $[0, 2]$ together with graphs of x and $x+x^2$.

Numerical work suggests the conjecture that for positive initial conditions (x, y)

$$y = L(x) \qquad \text{if and only if} \quad \gamma_e(x, y) = \gamma_o(x, y).$$

Computation tends to confirm this, since $\gamma_e(1, L(1)) \approx 1.88695\ 859$ and $\gamma_o(1, L(1)) \approx 1.88695\ 854$.

The conjecture is false. More positively, the conjecture is always true, but only to 6 or 7 decimal place accuracy! The values given above *are* correct to 8 decimal places. This can be verified with methods similar to what was

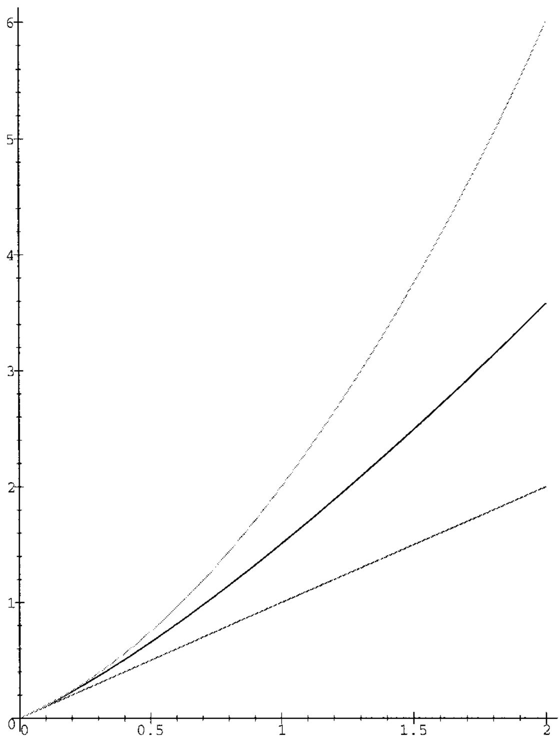

FIG. 3. Graph of L.

done in Example 2.1 combined with accurate computation of L as previously explained after Proposition 4.1, and therefore the conjecture is false.

Further explanation. Note that $\gamma_e(x, L(x))^{\sqrt{2}} = \gamma_o(L(x), x^2 + L(x)) = \gamma_o(L(x), L^{[2]}(x))$ etc., so that the ratio

$$R(x) = \frac{\log(\gamma_e(x, L(x)))}{\log(\gamma_o(x, L(x)))}$$

defined for $x > 0$ satisfies $R(L^{[2]}(x)) = R(x)$ and $R(L(x)) = \frac{1}{R(x)}$, and all of its values are attained on the interval $[1, L(L(1))]$, with $L(L(1)) \approx 2.501$. Figure 4 is a graph of this ratio. The scales of the vertical and horizontal axes are very different. The graph would be a horizontal line of height 1 if the conjecture above were correct. The upper and lower bounds of R are approximately Max $= 1.00000\ 01150$ (attained near $x_{\text{Max}} = 2.05$) and Min $= 0.99999\ 98850$ (attained near $x_{\text{Min}} = 1.28$). Of course, $\frac{1}{\text{Max}} \approx$ Min and $L(x_{\text{Min}}) \approx x_{\text{Max}}$.

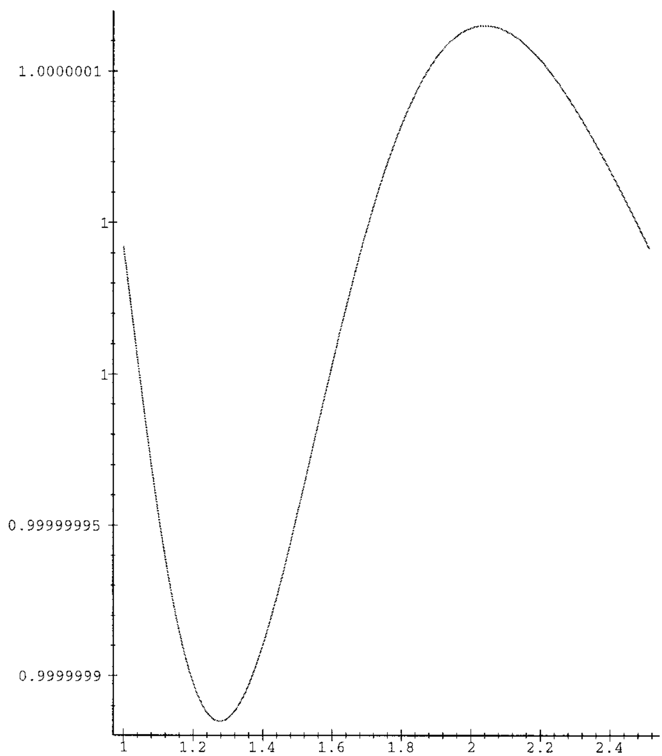

FIG. 4. Graph of R.

Figure 5 may be more useful. It shows several oscillations of R (x is in $[0.5, 4.5]$). R is a periodic function if one's "clock" is given by iterations of L. We use this comment implicitly below. Several other questions arise which we cannot answer now:

- Is there a simple condition in (γ_e, γ_o) space which is equivalent to $y = L(x)$?
- Suppose L^* is defined by requiring that the growth constants $\gamma_e(x, L^*(x))$ and $\gamma_o(x, L^*(x))$ agree. Does L^* have interesting properties?
- Is the map $(x, y) \to (\gamma_e(x, y), \gamma_o(x, y))$ a local diffeomorphism at every $(x, y) \in (0, \infty) \times (0, \infty)$?
- Is the map $(x, y) \to (\gamma_e(x, y), \gamma_o(x, y))$ a diffeomorphism of $(0, \infty) \times (0, \infty)$ onto $(1, \infty) \times (1, \infty)$?

We want to know if $(*)$ has a real analytic solution defined on all of $\mathbb{R}$. We show later that γ_e, γ_o, and L are positive real analytic functions away from 0 and have properties closely related to $(*)$. We use them to construct

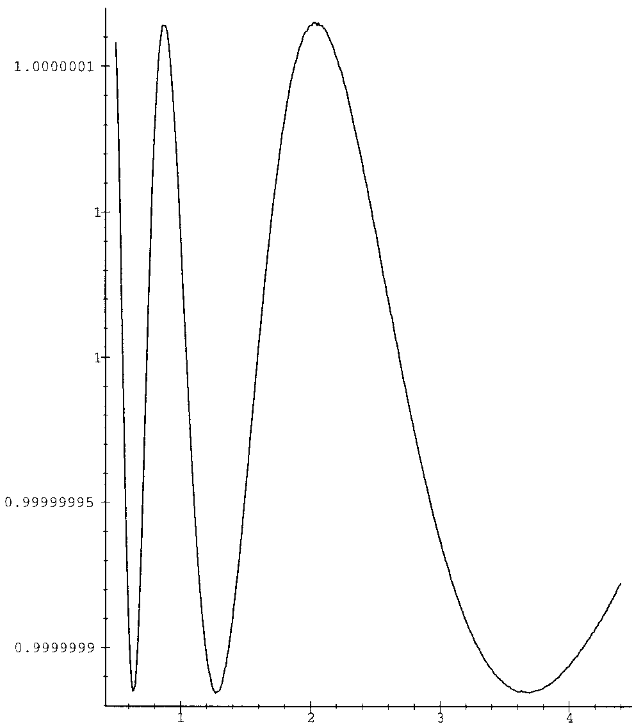

FIG. 5. Wider graph of R.

a solution. R's invariance under $L^{[2]}$ resembles properties of automorphic forms. Such invariant or nearly invariant functions are used in many ways.

Any solution to $(*)$ satisfies $f(x+n)=L^{[n]}(f(x))$, and, in effect, $(*)$ is a smooth interpolation of iterations of L applied to $f(x)$. For example, $L^{[17.4]}(f(5))$ "should be" $f(22.4)$.

Consider the function $g(t)=\gamma_e(t, L(t))$. Then $g(L^{[2]}(t))=\gamma_e(t, L(t))^2$ using the near-invariance of γ_e. We can see that if n is any positive integer, $g(L^{[2n]}(t))=(\gamma_e(t, L(t)))^{2^n}=(\gamma_e(t, L(t)))^{\sqrt{2}^{2n}}=g(t)^{\sqrt{2}^{2n}}$. This certainly suggests defining

$$S(x)=g^{[-1]}(g(t)^{\sqrt{2}^x})$$

and checking if it satisfies $(*)$ with $S(0)=t$.

This is not satisfactory. If $h(t)=\gamma_o(t, L(t))$, then $g(L(t))=h(t)^{\sqrt{2}}$. $S(x+1)$ is not likely to be the correct value since γ_o and γ_e are usually distinct at $(t, L(t))$. But $h(L(t))=g(t)^{\sqrt{2}}$ also. We reconsider.

Let $k(t)=g(t)\,h(t)$, so that $k(L(t))=g(L(t))\,h(L(t))=h(t)^{\sqrt{2}}\,g(t)^{\sqrt{2}}=k(t)^{\sqrt{2}}$. A proof by induction shows that $k(L^{[n]}(t))=k(t)^{\sqrt{2}^n}$, which leads to redefining S as

$$S(x)=k^{[-1]}(k(t)^{\sqrt{2}^x})$$

and checking that this definition of S satisfies $(*)$ with $S(0)=t$. We compute

$$\begin{aligned} L(S(x)) &= k^{[-1]}((k(S(x)))^{\sqrt{2}})=k^{[-1]}((k(k^{[-1]}(k(t)^{\sqrt{2}^x})))^{\sqrt{2}}) \\ &= k^{[-1]}((k(t)^{\sqrt{2}^x})^{\sqrt{2}})=k^{[-1]}(k(t)^{\sqrt{2}^{x+1}})=S(x+1) \end{aligned}$$

so that $L^{[2]}(S(x))=S(x+2)$, and $(*)$ is verified because L satisfies its functional equation. S intertwines L and translation by 1.

There is another detail to be checked. We have already remarked that γ_e and γ_e are real analytic on $(0, \infty)\times(0, \infty)$ (this will be proved in the next section) and that L is real analytic on $(0, \infty)$ (this will be shown in Section 8). Thus k must be real analytic on $(0, \infty)$. We have proved that $L'(s)>1$ for all $s>0$, and we shall show in the next section that

$$\frac{\partial\gamma_e(x, y)}{\partial x}>0, \qquad \frac{\partial\gamma_e(x, y)}{\partial y}\geqslant 0, \qquad \frac{\partial\gamma_o(x, y)}{\partial x}\geqslant 0, \qquad \text{and} \qquad \frac{\partial\gamma_o(x, y)}{\partial y}\geqslant 0$$

for all x, $y>0$. Therefore $k'(t)>0$ for all $t>0$, and we deduce that $k^{[-1]}$ is real analytic with domain $(1, \infty)$. It follows that $S(x)$ is a positive, increasing, real analytic solution defined on all of $\mathbb{R}$ and that S satisfies $(*)$.

We do not know how to characterize all real analytic solutions although some further information is given below. We do not know if there are non-constant real analytic solutions which can be 0. This seems unlikely since L is not analytic at 0 as we will see in Section 9. Real-valued solutions defined on $\mathbb{R}$ with negative values are not possible by Lemma 3.1. There certainly are C^∞ solutions on $\mathbb{R}$ which are sometimes 0: we will show that L is C^∞ on $[0, \infty)$ in Section 10. If we then take as initial condition any non-negative f which is C^∞ on $[0, 1]$, selected so that the formal Taylor series for f at 1 agrees with the formal Taylor series for $L(f)$ at 0 (any f which is 0 at both 0 and 1 and which is infinitely flat at both points has this property), we may extend f to all of $\mathbb{R}$ using iterations of L and obtain a solution of $(*)$ which is C^∞ on $\mathbb{R}$.

It may be useful at this point to contrast our recurrence with one that has been more widely studied. The classical Fibonacci recurrence is $H_{n+2} = H_{n+1} + H_n$. Two linearly independent solutions are $s_+(n) = r_+^n$ and $s_-(n) = r_-^n$ if $r_+ = (1+\sqrt{5})/2$ and $r_- = (1-\sqrt{5})/2$ respectively. The accompanying functional equation is $f(x+2) = f(x+1) + f(x)$. Since r_- is negative, s_- cannot be used to create a real-valued solution to the functional equation for all real x. But $s_+(x)$ is an entire solution to the functional equation. Corresponding to this is recognition that the only initial conditions (H_0, H_1) leading to doubly infinite sequences with constant sign which satisfy the Fibonacci recurrence are those which have $H_1 = r_+ H_0$. In fact, the function $V(x) = (r_+)\, x$ (just multiplication by r_+) and its interaction with the Fibonacci recurrence seem be quite analogous to L and its relationship to the QF recurrence. Thus only initial conditions $(x, V(x))$ for the Fibonacci recurrence always yield sequences $\{H_n\}_{n\in\mathbb{Z}}$ which satisfy the recurrence and for all $n \in \mathbb{Z}$ have $H_n = \lambda_n H_{n+1}$ with every $\lambda_n \in \mathbb{R}_{>0}$.

Any function f defined only on $[0, 1)$ can be used as initial conditions for a solution to the Fibonacci functional equation by using iterations of V to extend f's domain to all of $\mathbb{R}$: $f(x) = V^{[[x]]}(f(\{x\})) = r_+^{[x]} f(\{x\})$. Here the outer brackets in V's superscript refer to iteration, and the inner brackets, to "integer part", while $\{x\}$ means the fractional part of x. For example, $f(3.7)$ would be $((1+\sqrt{5})/2)^3 f(0.7)$.

Another way to create solutions begins with m, a periodic function with period 1. Then

$$F(x) = r_+^x\, m(x)$$

solves the Fibonacci functional equation for two reasons. First, V satisfies an appropriate auxiliary functional equation,

$$V(V(x)) = V(x) + x$$

and second, V intertwines translation with iteration on functions like F: $V(F(x)) = F(x+1)$. F inherits smoothness from m. Thus if m is real analytic and periodic of period 1 on $\mathbb{R}$, F must be real analytic on $\mathbb{R}$. Also, if m is an entire periodic function of period 1, F must be an entire function satisfying the Fibonacci functional equation.

Results for the QF recurrence are similar to these. We use L in place of V and create an appropriate class of functions. Indeed, if m is periodic of period 1 and positive, then

$$F(x) = k^{[-1]}(k(t)^{m(x)\sqrt{2}^x})$$

is a solution of $(*)$. This again follows from L's functional equation and L's intertwining of translation with iteration on such F's. The latter can be checked by direct computation, as was done earlier with S. It seems likely that all everywhere-positive solutions of $(*)$ defined on $\mathbb{R}$ arise this way. If m is real analytic, so is F. We do not know if there are nonconstant entire solutions to $(*)$. The behavior of candidates for m when F is one of the previously described C^∞ solutions is not clear when such solutions have zeros.

THEOREM 4.3. *Suppose* $k(t) = \gamma_e(t, L(t))\, \gamma_o(t, L(t))$ *for* $t > 0$, *and* m *is a positive periodic function of period* 1. *Then* k *maps* $(0, \infty)$ *to* $(1, \infty)$, *and* $F(x) = k^{[-1]}(k(t)^{m(x)\sqrt{2}^x})$ *satisfies* $(*)$. *If* m *is* C^k *or real analytic, so is* F.

Proof. The only part of this which remains to be verified is that k maps $(0, \infty)$ onto $(1, \infty)$. But $k(\tau) > 0$ and $k(L^{[n]}(\tau)) = k(\tau)^{\sqrt{2}^n}$ for all $n \in \mathbb{Z}$. We know that $\lim_{n\to\infty} L^{[n]}(\tau) = \infty$ and $\lim_{n\to-\infty} L^{[n]}(\tau) = 0$. Also, k is continuous, so the Intermediate Value Theorem applies to show that all of $(1, \infty)$ is in the range of k. ∎

5. COMPLEX SEQUENCES AND REAL ANALYTICITY

If $(w, z) \in \mathbb{C}^2$ let $\|(w, z)\| = \max\{|w|, |z|\}$. Let $\Phi(w, z) = (z + w^2, z + w^2 + z^2)$, and let $\Phi^{[n]} = \Phi \circ \Phi \circ \cdots \circ \Phi$ (composition n times). We consider again the recurrence $H_{n+2} = H_{n+1} + H_n^2$ with initial conditions $H_0 = w$ and $H_1 = z$. If $(w_n, z_n) = \Phi^{[n]}(w, z)$, then $w_n = H_{2n}$ and $z_n = H_{2n+1}$. We establish some estimates analogous to the first lemmas of this paper. The function $x/(x+1)$ used there is replaced by $g(x) = x(x+1)/(1-x)^2$. g is increasing on $[0, 1)$, $g(0) = 0$ and $g(\frac{1}{3}) = 1$.

If $z_n \neq 0$, let $c_n = w_{n+1}/z_n^2$. If $w_{n+1} \neq 0$, let $d_n = z_{n+1}/w_{n+1}^2$. Finally, let $k_n = \|(c_n, d_n)\|$ when both c_n and d_n exist.

THEOREM 5.1. *Suppose that* c_N *and* d_N *are defined and* $k_N < \frac{1}{3}$. *Then for all* $n \geqslant N$, c_n, d_n, *and* k_n *are defined, and* $k_{n+1} \leqslant \kappa k_n$ *with* $\kappa = g(k_N) < 1$ *so that* $\lim_{n \to \infty} k_n = 0$. *Finally, if* $k_M \leqslant \frac{1}{4}$, *then* $k_{n+1} \leqslant \frac{20}{9} k_n^2$ *for all* $n \geqslant M$.

Proof. Suppose $|w_{n+1}| \leqslant k_n \, |z_n|^2$ and $|z_{n+1}| \leqslant k_n \, |w_{n+1}|^2$ for some $k_n < \frac{1}{3}$. Then

$$|w_{n+2}| = |z_{n+1} + w_{n+1}^2| \leqslant |z_{n+1}| + |w_{n+1}|^2 \leqslant (k_n + 1) \, |w_{n+1}|^2$$

and

$$|z_{n+1}| = |w_{n+1} + z_n^2| = |w_{n+1}| \left| 1 + \frac{z_n^2}{w_{n+1}} \right| \geqslant |w_{n+1}| \left(\frac{1}{k_n} - 1 \right).$$

Since $k_n \leqslant \frac{1}{3}$, $z_{n+1} \neq 0$.

Therefore

$$\frac{|w_{n+2}|}{|z_{n+1}|^2} \leqslant \frac{(k_n + 1) \, k_n^2}{(1 - k_n)^2} = \left(\frac{(k_n + 1) \, k_n}{(1 - k_n)^2} \right) k_n = g(k_n) \, k_n$$

so that $|w_{n+2}|/|z_{n+1}|^2 = |c_{n+1}| < k_n$ must hold.

Now we know that $|w_{n+2}| \leqslant k_n \, |z_{n+1}|^2$ and $|z_{n+1}| \leqslant k_n \, |w_{n+1}|^2$. Paralleling the proof of Lemma 1, the argument given just previously shows that $|z_{n+2}|/|w_{n+2}|^2 = |d_{n+1}| < k_n$.

We now know $k_{n+1} = \max\{|c_{n+1}|, |d_{n+1}|\} < k_{n+1}$. More precisely we have shown that $k_{n+1} \leqslant g(k_n) \, k_n$ when $k_n < \frac{1}{3}$. If we have N with $g(k_N) < \frac{1}{3}$ and if $\kappa = g(k_N)$ then $\kappa < 1$ and $k_{n+1} \leqslant \kappa k_n$ for all $n \geqslant N$.

Since $g(x) = x(x+1)/(1-x)^2 = ((x+1)/(1-x)^2) \, x$ can be written as a product of two increasing functions, $g(x) \leqslant \frac{20}{9} x$ for $x \in [0, \frac{1}{4}]$. The last assertion of the theorem is now clear. ∎

Suppose $(x_0, y_0) \in \mathbb{R}^2$. If $H_{n+2} = H_{n+1} + H_n^2$ with $H_0 = x_0$ and $H_1 = y_0$. We separate the even and odd subsequences of $\{H_n\}$. Let $x_n = H_{2n}$ and $y_n = H_{2n+1}$. Define Q to be $\{(x, y): x \geqslant 0, \; y \geqslant 0, \; (x, y) \neq (0, 0)\}$ in $\mathbb{R}^2$. If $(x_0, y_0) \in Q$, then Corollary 2.1 provides constants $\alpha(x_0, y_0) = \gamma_e(x_0, y_0) > 1$ and $\beta(x_0, y_0) = (\gamma_o(x_0, y_0))^{\sqrt{2}} > 1$ so that $x_n/(\alpha(x_0, y_0))^{2^n} \to 1$ and $y_n/(\beta(x_0, y_0))^{2^n} \to 1$ as $n \to \infty$. We also know from Theorem 2.1 that if $c_n = x_{n+1}/y_n^2$ and $d_n = y_{n+1}/x_{n+1}^2$ then $c_n \to 0$ and $d_n \to 0$ as $n \to \infty$.

THEOREM 5.2. *The maps* $\alpha(x, y)$ *and* $\beta(x, y)$ *can be extended to an open neighborhood* U *of* Q *in* $\mathbb{R}^2$. $\alpha(x, y)$ *and* $\beta(x, y)$ *are real analytic and non-zero on* U, *and* $\lim_{n \to \infty}(x_n/(\alpha(x_0, y_0))^{2^n}) = 1$ *and* $\lim_{n \to \infty}(y_n/(\beta(x_0, y_0))^{2^n}) = 1$ *for all* $(x, y) \in U$.

Proof. Suppose $(x_0, y_0) \in Q$. We show that α and β can be defined in an open neighborhood of (x_0, y_0) by realizing that they are uniform limits of holomorphic functions in an open ball centered at $(x_0, y_0) \in \mathbb{C}^2$. The notation of the discussion and proof of Theorem 5.1 is used.

By Theorem 2.1, there is N so that $x_{N+1}/y_N^2 < \frac{1}{4}$ and $y_{N+1}/x_{N+1}^2 < \frac{1}{4}$. If $(w, z) \in \mathbb{C}^2$, then (w_n, z_n) is a pair of polynomial functions of w and z. Let $B_\delta = B_\delta(x_0, y_0) = \{(w, z) \in \mathbb{C}^2 : |w - x_0| < \delta, |z - y_0| < \delta\}$. Note that c_n and d_n are defined at (x_0, y_0) for all n. Select δ small enough so that $c_1, c_2, ..., c_N$ and $d_1, d_2, ..., d_N$ are defined in all of B_δ. This is possible since these functions are all defined at points where certain polynomial functions are non-zero, and (x_0, y_0) is such a point. We can also select δ small enough so that $c_N < \frac{1}{4}$ and $d_N < \frac{1}{4}$ for all $(w, z) \in B_\delta$ since c_N and d_N are continuous functions where they are defined. We may additionally assume that $\operatorname{Re} z_N > 0$ and $\operatorname{Re} w_N > 0$ in all of B_δ by again shrinking δ if necessary.

All the hypotheses of Theorem 5.1 are valid for all $(w, z) \in B_\delta$. Since $k_{n+1} \leqslant \frac{20}{9} k_n^2$ and $k_N < \frac{1}{4}$, the two sequences of rational functions $\{c_n\}$ and $\{d_n\}$ converge uniformly to 0 in B_δ as $n \to \infty$.

For $n > N$, define

$$a_n = \frac{1}{2^N} \operatorname{Log}(z_N) + \sum_{j=N}^{n-1} \frac{1}{2^{j+1}} \operatorname{Log}(1 + c_j)$$

$$b_n = \frac{1}{2^N} \operatorname{Log}(w_N) + \sum_{j=N}^{n-1} \frac{1}{2^{j+1}} \operatorname{Log}(1 + d_j),$$

where Log is the principal branch of log, defined here by $\operatorname{Log}(z) = \log(|z|) + i \arg z$ with $-\pi < \arg z \leqslant \pi$. Since $\operatorname{Re} z_N > 0$, $\operatorname{Re} w_N > 0$, $|c_j| < \frac{1}{4}$, and $|d_j| < \frac{1}{4}$, the functions a_n and b_n are all defined and holomorphic in B_δ.

This definition is motivated by the following considerations. If $n > N$, $z_{n+1} = z_n + w_n^2 + z_n^2 = w_{n+1}^2 + z_n^2 = (1 + (w_{n+1}/z_n^2))\, z_n^2 = (1 + c_n)\, z_n^2$. Therefore, $(z_{n+1})^{2^{-(n+1)}}$ "should be" $(1 + c_n)^{2^{-(n+1)}} (z_n)^{2^{-n}} = (1 + c_n)^{2^{-(n+1)}} (1 + c_{n-1})^{2^{-n}} (z_{n-1})^{2^{-(n-1)}} = \cdots$ "down to" N. Of course, holomorphic roots are problematic, but these equations can be made precise with logs on correctly restricted domains, as done above. A similar result is true for w_{n+1} and d_n.

Elementary estimates for log show that $|\operatorname{Log}(1 + c_j)| \leqslant 2\, |c_j| < \frac{1}{2}$ and $|\operatorname{Log}(1 + d_j)| \leqslant 2\, |d_j| < \frac{1}{2}$ when $j \geqslant N$, so the two sequences of holomorphic functions $\{a_n\}$ and $\{b_n\}$ converge uniformly on B_δ to holomorphic limits a and b.

Define $\alpha_n = \exp(a_n)$ and $\beta_n = \exp(b_n)$. An induction proof combined with the algebra above shows that $z_n = \alpha_n^{2^n}$ and $w_n = \beta_n^{2^n}$. Finally, we define $\alpha = \exp a$ and $\beta = \exp b$.

The theorem's final claim is verified if we show that $\lim_{n \to \infty}(x_n/\alpha(x_0, y_0)^{2^n}) = 1$, with a similar statement and proof for y_n and β. This limit is equivalent to $\lim_{n\to\infty}(\alpha_n^{2^n}/\alpha(x_0, y_0)^{2^n})=1$. On the "log level" we must show $\lim_{n\to\infty} 2^n \sum_{j=n}^{\infty}(1/2^{j+1})\,\mathrm{Log}(1+c_j)=0$. The quadratic convergence of $\{k_n\}$ in Theorem 5.1 guarantees this. ▮

COROLLARY 5.1. *The following equations hold for all* (x, y) *in* Q:

$$\alpha(x^2+y, x^2+y^2+y)=\alpha(x, y)^2 \qquad \text{and}$$

$$\beta(x^2+y, x^2+y^2+y)=\beta(x, y)^2;$$

$$\alpha(x, y)^2=\beta(y, x^2+y) \qquad \text{and}$$

$$\alpha(y, x^2+y)=\beta(x, y).$$

Proof. The unique growth constants for our recurrence given initial conditions $(x_0, y_0)\in U$ are $\alpha(x_0, y_0)$ and $\beta(x_0, y_0)$. If $(x_0, y_0)\in Q$ then the next two terms produced by the recurrence are $(x_0^2+y_0, x_0^2+y_0^2+y_0)$. Shifting the index in the limit of the previous theorem shows that $\alpha(x_0, y_0)^2$ and $\beta(x_0, y_0)^2$ are growth constants for the sequence with those terms as initial conditions, which proves the equations above.

The second set of assertions about α and β are obtained by shifting the index only one step. Of course the first pair of equations can be deduced from the second by applying them once each. ▮

We hope that these functional equations will allow us to analyze other properties of our sequences, such as which polynomial identities are satisfied by generic sequences resulting from applying the recurrence.

We need to analyze the partial derivatives of γ_e and γ_o in order to complete the proof of Theorem 4.3.

For x and y positive, we let $x_1=y+x^2$ and $y_1=x_1+y^2$, and generally for any $k\geqslant 1$ we define $x_{k+1}=y_k+x_k^2$ and $y_{k+1}=x_{k+1}+y_k^2$. Certainly $x_k=F_k(x, y)$ and $y_k=G_k(x, y)$ where F_k and G_k are polynomials with positive integral coefficients. For fixed $\tilde{x}$ and $\tilde{y}$ positive we have just shown that there exists $\delta>0$ so that if $\|(w, z)-(\tilde{x}, \tilde{y})\|<\delta$, the maps $(x, y)\to (F_k(x, y))^{2^{-k}}$ and $(x, y)\to (G_k(x, y))^{2^{-k}}$ (initially defined only for positive real x and y) extend to holomorphic maps which we denote $(F_k(w, z))^{2^{-k}}$ and $(G_k(w, z))^{2^{-k}}$. Furthermore, the maps $\{(F_k(w, z))^{2^{-k}}\}_{k\in\mathbb{N}}$ converge uniformly on $B_\delta(\tilde{x}, \tilde{y})=\{(w, z)\colon \|(w, z)-(\tilde{x}, \tilde{y})\|<\delta\}$ to a holomorphic map $\alpha(w, z)=\gamma_e(w, z)$, with a similar statement for $\{(G_k(w, z))^{2^{-k}}\}_{k\in\mathbb{N}}$ and $\beta(w, z)=(\gamma_o(w, z))^{\sqrt{2}}$. Certainly for any positive $\delta'<\delta$,

$$\frac{\partial}{\partial w}((F_k(w, z))^{2^{-k}})\to\frac{\partial}{\partial w}\gamma_e(w, z) \qquad \text{and} \qquad \frac{\partial}{\partial z}((F_k(w, z))^{2^{-k}})\to\frac{\partial}{\partial z}\gamma_e(w, z)$$

as $k\to\infty$, uniformly for (w, z) satisfying $\|(w, z)-(\tilde{x}, \tilde{y})\|<\delta'$.

We only need information about the first partial derivatives of $\gamma_e(x, y)$ and $\gamma_o(x, y)$ for real positive x and y. Since larger initial conditions lead to larger growth constants, $x \to \gamma_e(x, y)$ and $y \to \gamma_e(x, y)$ are increasing maps. Thus $(\partial\gamma_e/\partial x)(x, y) \geqslant 0$ and $(\partial\gamma_e/\partial y)(x, y) \geqslant 0$ for real positive x and y. Similar statements are true for γ_o.

PROPOSITION 5.1. *If $x > 0$ and $y > 0$, then $(\partial\gamma_e/\partial x)(x, y) > 0$.*

Proof. Take logarithms in the first limit displayed above. Thus it will be sufficient to prove, given x and y positive, there is $c > 0$ so that

$$\frac{\partial}{\partial x}(2^{-k}\log F_k(x, y)) = 2^{-k}\frac{\dfrac{\partial}{\partial x}F_k(x, y)}{F_k(x, y)} \geqslant c.$$

Let $x_k = F_k(x, y)$ and $y_k = G_k(x, y)$ and note that $\partial y_j/\partial x \geqslant 0$ for all $j \geqslant 0$. Then

$$2^{-k}\frac{\dfrac{\partial}{\partial x}F_k(x, y)}{F_k(x, y)} = 2^{-k}\frac{\left(2x_{k-1}\dfrac{\partial x_{k-1}}{\partial x} + \dfrac{\partial y_{k-1}}{\partial x}\right)}{x_{k-1}^2 + y_{k-1}} \geqslant 2^{-(k-1)}\frac{x_{k-1}\dfrac{\partial x_{k-1}}{\partial x}}{x_{k-1}^2 + y_{k-1}}$$

$$\geqslant 2^{-(k-1)}\frac{x_{k-1}\dfrac{\partial}{\partial x}(x_{k-2}^2 + y_{k-2})}{x_{k-1}^2 + y_{k-1}}$$

$$\geqslant 2^{-(k-2)}\frac{x_{k-1}x_{k-2}\dfrac{\partial x_{k-2}}{\partial x}}{x_{k-1}^2 + y_{k-1}},$$

where we have used $\partial y_{k-1}/\partial x \geqslant 0$ and $\partial y_{k-2}/\partial x \geqslant 0$. We may continue and obtain

$$2^{-k}\frac{\dfrac{\partial}{\partial x}F_k(x, y)}{F_k(x, y)} \geqslant \frac{x_{k-1}x_{k-2}\cdots x_0}{x_{k-1}^2 + y_{k-1}},$$

where $x_0 = x$.

If $d_j = y_j/x_j^2$, then we know (Section 2) that $\lim_{j\to\infty} d_j = 0$ and $d_{j+1} \leqslant d_j^2$ for j large. Then we may write $x_j = y_{j-1} + x_{j-1}^2 = (1 + d_{j-1})\, x_{j-1}^2$ or $x_{j-1} = x_j^{1/2}/(1 + d_{j-1})^{1/2}$ which leads to

$$x_{k-2} = \frac{x_{k-1}^{1/2}}{(1 + d_{k-2})^{1/2}}, \quad x_{k-3} = \frac{x_{k-2}^{1/2}}{(1 + d_{k-3})^{1/2}} = \frac{x_{k-1}^{1/2^2}}{(1 + d_{k-2})^{1/2}\,(1 + d_{k-3})^{1/2}}, \ldots$$

so we have generally

$$x_{k-j}=\frac{x_{k-1}^{1/2^{j-1}}}{(1+d_{k-2})^{2^{-j+1}}\,(1+d_{k-3})^{2^{-j+2}}\cdots(1+d_{k-j})^{2^{-1}}}.$$

It follows that the product $x_0 x_1 \cdots x_{k-1}$ can be underestimated by

$$\begin{aligned} &x_{k-1}^{1+1/2+1/2^2+\cdots+1/2^{k-1}}(1+d_{k-2})^{-1/2-1/2^2-\cdots-1/2^{k-1}} \\ &\qquad\times(1+d_{k-3})^{-1/2-1/2^2-\cdots-1/2^{k-1}}\cdots(1+d_0)^{-1/2} \\ &\qquad\geqslant x_{k-1}^2\big(x_{k-1}^{2^{-(k-1)}}\big)^{-1}\left(\frac{1}{(1+d_0)(1+d_1)\cdots(1+d_{k-2})}\right). \end{aligned}$$

Since $x_{k-2}^2+y_{k-1}=(1+d_{k-1})\,x_{k-1}^2$, we see that

$$\begin{aligned} 2^{-k}\,\frac{\dfrac{\partial}{\partial x}F_k(x,y)}{F_k(x,y)} &\geqslant \frac{x_{k-1}x_{k-2}\cdots x_0}{x_{k-1}^2+y_{k-1}} \\ &\geqslant \big(x_{k-1}^{2^{-(k-1)}}\big)^{-1}\left(\frac{1}{(1+d_0)(1+d_1)\cdots(1+d_{k-2})(1+d_{k-1})}\right) \end{aligned}$$

This last expression is sufficiently simple to underestimate effectively. Our estimates imply that $\sum_{j=0}^{\infty} d_j=D<\infty$, so $\lim_{k\to\infty}\prod_{j=0}^{k-1}(1+d_j)=\kappa>0$. We also know that $\lim_{k\to\infty} x_{k-1}^{2^{-(k-1)}}=\gamma_e(x,y)$, where $1<\gamma_e(x,y)<\infty$. Therefore

$$\liminf_{k\to\infty} 2^{-k}\,\frac{\dfrac{\partial}{\partial x}F_k(x,y)}{F_k(x,y)}\geqslant\left(\frac{1}{\gamma_e(x,y)}\right)\left(\frac{1}{\kappa}\right)>0$$

and we are done. ∎

In Theorem 5.2 we have shown that the functions α and β extend analytically to an open neighborhood of Q in $\mathbb{C}^2$. In general, given z_0 and z_1 in $\mathbb{C}$, suppose that $\{z_j\}$, $j\geqslant 0$, satisfies the QF recurrence and define $v_j=z_{2j}$ and $w_j=z_{2j+1}$. Thus, if we define $F=\Psi^{[2]}$, $(v_n, w_n):=F^{[n]}(v_0, w_0)$. If $\mathscr{S}$ denotes the set of stable initial conditions in $\mathbb{C}^2$ (so $(w,z)\in\mathscr{S}$ if and only if $\sup_n\{\|\Psi^{[n]}(w,z)\|\}<\infty$), then for $(v_0,w_0)\in\mathbb{C}^2$ and $(v_0,w_0)\notin\mathscr{S}$, one can ask whether there exist complex numbers $\alpha(v_0,w_0)$ and $\beta(v_0,w_0)$ such that

$$\lim_{n\to\infty}\frac{v_n}{\alpha(v_0,w_0)^{2^n}}=1 \qquad\text{and}\qquad \lim_{n\to\infty}\frac{w_n}{\beta(v_0,w_0)^{2^n}}=1.$$

Less generally, one can ask whether there exist positive real numbers $a(v_0, w_0)$ and $b(v_0, w_0)$ such that

$$\lim_{n\to\infty} \frac{|v_n|}{a(v_0, w_0)^{2^n}} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{|w_n|}{b(v_0, w_0)^{2^n}} = 1.$$

One can also ask whether only some of these limits exist or whether there exists a positive, real number $c(v_0, w_0)$ such that

$$\lim_{n\to\infty} \frac{\|(v_n, w_n)\|}{c(v_0, w_0)^{2^n}} = 1.$$

Finally, one can ask whether, for $(v_0, w_0) \notin \mathscr{S}$, the following limit exists:

$$\lim_{n\to\infty} \frac{\log(\|(v_n, w_n)\|)}{2^n} := \lim_{n\to\infty} G_n(v_0, w_0) := G(v_0, w_0).$$

We shall sketch here some answers to these questions, but we shall defer proofs to a future paper. In the next section it is proved (see Lemma 6.1) that $\mathscr{S}$ is a compact set which is contained in the closed ball of radius 2 in $\mathbb{C}^2$. Also, it is not hard to prove that, for $(v_0, w_0) \notin \mathscr{S}$, $\sup_n\{G_n(v_0, w_0)\} < \infty$.

LEMMA 5.1. *Assume that* $(v_0, w_0) \in \mathbb{C}^2$, $(v_0, w_0) \notin \mathscr{S}$ *and* $k > (\frac{1}{2})(1+\sqrt{5})$. *Suppose also that there exists a sequence of integers* $n_i \to \infty$ *such that* $v_{n_i} \neq 0$ *and* $|(w_{n_i}/v_{n_i})| \geq k$. *Then there exists an integer* N_k *such that* $v_n \neq 0$ *and* $|(w_n/v_n)| \geq k$ *for all* $n \geq N_k$. *It follows that if* $\limsup_{n\to\infty} |(w_n/v_n)| = \infty$, *then* $\lim_{n\to\infty} |(w_n/v_n)| = \infty$.

LEMMA 5.2. *Assume that* $(v_0, w_0) \in \mathbb{C}^2$, $(v_0, w_0) \notin \mathscr{S}$ *and* $\limsup_{n\to\infty} |(w_n/v_n)| = \infty$. *Then there exists a real number* $b := b(v_0, w_0) > 1$ *such that*

$$\lim_{n\to\infty} \frac{|w_n|}{b^{2^n}} = 1.$$

LEMMA 5.3. *Assume that* $(v_0, w_0) \in \mathbb{C}^2$, *that* $(v_0, w_0) \notin \mathscr{S}$ *and that there exists a constant* M *such that* $|(w_n/v_n)| \leq M$ *for all large* n. *Then there exists a real number* $a := a(v_0, w_0) > 1$ *such that*

$$\lim_{n\to\infty} \frac{|v_n|}{a^{2^n}} = 1.$$

The following theorem was stated by the referee in his report.

THEOREM 5.3. *For all* $(v_0, w_0) \in \mathbb{C}^2$, $(v_0, w_0) \notin \mathcal{S}$, *the number* $G(v_0, w_0)$ *(defined above) exists and is a finite, positive number. The map* $(v_0, w_0) \to G(v_0, w_0)$ *is plurisubharmonic.*

We now turn to the question of whether, given $(v_0, w_0) \in \mathbb{C}^2$, $(v_0, w_0) \notin \mathcal{S}$, there exist $\alpha := \alpha(v_0, w_0)$ and $\beta := \beta(v_0, w_0)$ such that

$$\lim_{n\to\infty} \frac{v_n}{\alpha(v_0, w_0)^{2^n}} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{w_n}{\beta(v_0, w_0)^{2^n}} = 1.$$

We define $\mathcal{A}$ to be the the set of $(v_0, w_0) \in \mathbb{C}^2$, $(v_0, w_0) \notin \mathcal{S}$, such that $\alpha(v_0, w_0)$ and $\beta(v_0, w_0)$ as above exist.

THEOREM 5.4. *Assume that* $(v_0, w_0) \in \mathbb{C}^2$ *and* $(v_0, w_0) \notin \mathcal{S}$. *It follows that* $(v_0, w_0) \in \mathcal{A}$ *if and only if there exists an integer* $N \geqslant 1$ *such that* $v_N \neq 0$, $|(w_N/v_N)| > 2$ *and* $|(w_N/v_N^2)| < (1/3)$. *Given* $(v_0^*, w_0^*) \in \mathcal{A}$, *there exists* $\delta > 0$ *such that for all* $(v_0, w_0) \in B_\delta(v_0^*, w_0^*)$ *(where* $B_\delta(v_0^*, w_0^*)$ *denotes the open ball in* $\mathbb{C}^2$ *of radius* δ *and center* (v_0^*, w_0^*)*) one has* $(v_0, w_0) \in \mathcal{A}$. *The numbers* $\alpha(v_0, w_0)$ *and* $\beta(v_0, w_0)$ *can be selected so that the maps* $(v_0, w_0) \to \alpha(v_0, w_0)$ *and* $(v_0, w_0) \to \beta(v_0, w_0)$ *are holomorphic maps defined on* $B_\delta(v_0^*, w_0^*)$. *If* $\alpha_1(v_0, w_0)$ *and* $\beta_1(v_0, w_0)$ *are holomorphic maps defined on* $B_\delta(v_0^*, w_0^*)$ *and such that*

$$\lim_{n\to\infty} \frac{v_n}{\alpha_1(v_0, w_0)^{2^n}} = 1 \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{w_n}{\beta_1(v_0, w_0)^{2^n}} = 1,$$

then there are complex numbers u_0 *and* u_1 *such that* $\alpha_1(v_0, w_0) = u_0 \alpha(v_0, w_0)$ *and* $\beta_1(v_0, w_0) = u_1 \beta(v_0, w_0)$ *for all* $(v_0, w_0) \in B_\delta(v_0^*, w_0^*)$. *The constants* u_j, $j = 0, 1$, *satisfy* $\lim_{n\to\infty} u_j^{2^n} = 1$.

6. DEGREE THEORY AND PERIODIC POINTS

Recall that $\Psi(w, z) = (z, z + w^2)$. As before, $\Psi^{[n]} = \Psi \circ \Psi \circ \cdots \circ \Psi$ (composition n times). We define $\mathcal{S}$, the set of *stable initial conditions* of the recurrence $H_{n+2} = H_{n+1} + H_n^2$, by the following:

$$\mathcal{S} = \{(w, z) \in \mathbb{C}^2 : \sup_n \{\|\Psi^{[n]}(w, z)\|\} < \infty\}.$$

$\mathcal{S}$ must contain any *periodic points* of the recurrence: those (w, z) for which there is an N with $\Psi^{[N]}(w, z) = (w, z)$. Considerations from degree theory will show that $\mathcal{S}$ contains infinitely many periodic points.

If $(w, z) \in \mathbb{C}^2$, then we can always go backwards: there are complex numbers q with $q^2 + w = z$. If $\|(w, z)\| = R$ then $|q| \leqslant \sqrt{2R}$, so when $R \leqslant 2$, all ancestors of (w, z) will be in $\overline{B_R(0, 0)}$. The lemma following shows that sup in the definition above can be taken over all integers or only over positive integers: the set $\mathscr{S}$ will be the same.

If $0 \leqslant t \leqslant 1$, define $\Psi_t \colon \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\Psi_t(w, z) = (z, tz + w^2).$$

Of course $\Psi_1 = \Psi$. The composition of Ψ_t with itself n times will be denoted $\Psi_t^{[n]}$.

Lemma 6.1. *If $\|(w, z)\| = R \geqslant 2$ and $t \in [0, 1]$, then $\|\Psi_t(w, z)\| \geqslant \|(w, z)\|$. If $R > 2$, there exists a unique positive number $c = c(R)$ such that $c^2 R - 1 = \frac{1}{c}$ and $c < \sqrt{\frac{1}{R} + \frac{1}{2}} < 1$ so that if $\|(w, z)\| \geqslant R$, then $\|\Psi_t^{[2]}(w, z)\| \geqslant (\frac{1}{c})\, \|(w, z)\|$ and $\lim_{n \to \infty} \|\Psi_t^{[n]}(w, z)\| = \infty$.*

Proof. Suppose that $\|(w, z)\| = R \geqslant 2$. If $|w| = R$ and $0 \leqslant t \leqslant 1$ then

$$\|\Psi_t(w, z)\| \geqslant |w^2 + tz| \geqslant |w|^2 - t\,|z| \geqslant R^2 - tR \geqslant R(R-1) \geqslant R.$$

If $|w| < R$, then $|z| = R$, so $\|\Psi_t(w, z)\| \geqslant R$. Therefore $\|\Psi_t(w, z)\| \geqslant \|(w, z)\|$.

If $R > 2$, we define $\psi_1(c) = c^2 R - 1$ and $\psi_2(c) = \frac{1}{c}$ for $c > 0$. ψ_1 is strictly increasing on $(0, \infty)$ and ψ_2 is strictly decreasing on $(0, \infty)$. Since $\lim_{c \to 0^+} \psi_2(c) = \infty > \lim_{c \to 0^+} \psi_1(c) = -1$ and $\psi_2(1) = 1 < \psi_1(1) = R - 1$, the Intermediate Value Theorem shows that there is a unique $c = c(R) \in (0, 1)$ with $\psi_1(c) = \psi_2(c)$. The Intermediate Value Theorem also implies that if $t \in (0, 1)$ satisfies $\psi_1(t) > \psi_2(t)$, then $c(R) < t$. We select $t_R = \sqrt{\frac{1}{R} + \frac{1}{2}}$ and consider the following functions of R: $(\psi_1(t_R))^2 = R^2/4$ and $(\psi_2(t_R))^2 = 2R/R + 2$. These functions are both 1 when $R = 2$. Differentiation shows that both functions increase for $R > 2$, but $(\psi_1(t_R))^2$ is concave up and $(\psi_2(t_R))^2$ is concave down as functions of R for $R > 2$. At $R = 2$, the first derivative of the former is 1 and the first derivative of the latter is $\frac{1}{4}$. We have verified that $\psi_1(t_R) > \psi_2(t_R)$, so $c(R) < t_R = \sqrt{\frac{1}{R} + \frac{1}{2}}$.

Our previous remarks show that if $\|(w, z)\| \geqslant 2$, then $\{\|\Psi_t^{[n]}(w, z)\|\}_{n \in \mathbb{N}}$ is an increasing sequence. If we can prove that

$$\|\Psi_t^{[2]}(w, z)\| \geqslant \lambda\, \|(w, z)\|$$

whenever $\|(w, z)\| \geqslant R > 2$ where $\lambda = 1/c(R) > 1$, then $\|\Psi_t^{[2n]}(w, z)\| \geqslant \lambda^n\, \|(w, z)\|$, so that $\lim_{n \to \infty} \|\Psi_t^{[n]}(w, z)\| = \infty$ for such (w, z)'s.

We now prove the needed inequality for $\|\Psi_t^{[2]}(w, z)\|$ when $\|(w, z)\| = S \geqslant R > 2$. There are several cases. First, if $|w| \geqslant cS$ where $c = c(R)$,

$$\|\Psi_t(w, z)\| = |w^2 + tz| \geqslant c^2 S^2 - tS = S(c^2 S - t) \geqslant S(c^2 R - 1) = S\left(\frac{1}{c}\right) = \lambda S.$$

Then $\|\Psi_t^{[2]}(w, z)\| \geqslant \|\Psi_t(w, z)\| \geqslant \lambda \, \|(w, z)\|$.

Alternatively, suppose $|w| < cS < S$. Then $|z|$ must be S. We consider $(w_1, z_1) = \Psi_t(w, z) = (z, w^2 + tz)$ and let $S_1 = \|(w_1, z_1)\|$. If $z_1 = w^2 + tz$ has modulus at least λS, then $S_1 \geqslant \lambda S$, and again $\|\Psi_t^{[2]}(w, z)\| \geqslant \|\Psi_t(w, z)\| \geqslant \lambda \, \|(w, z)\|$.

So we now consider $|w| < cS$ and $|z_1| < \lambda S$. We know that $S_1 \geqslant S$ and $|w_1| = |z| = S < \lambda S$. Then $|w_1| = c(\lambda S) > cS_1$, and the point (w_1, z_1) exactly satisfies the hypothesis for the first case discussed. It follows that $\|\Psi_t(w_1, z_1)\| = \|\Psi_t^{[2]}(w, z)\| \geqslant \lambda S_1 \geqslant \lambda S$. ∎

Ψ itself need not increase norm on a closed bidisc centered at $(0, 0)$ of radius $R > 0$, since $\|\Psi(\alpha i, R)\| = \|(R, R - \alpha^2)\| = R$ for α real with $0 \leqslant \max(|\alpha|, \alpha^2) \leqslant R$. The lemma shows that the set of stable initial conditions must be bounded: surely $\mathscr{S} \subset \overline{B_2(0, 0)}$. We do not know $R_{\mathscr{S}} = \sup\{\|(w, z)\| : (w, z) \in \mathscr{S}\}$. One can show (see Section 7) that Ψ has a periodic point ζ of period 4 with $\|\zeta\| > 1.7$ and a refinement of Lemma 6.1 shows that $R_{\mathscr{S}} < 2$, so we have $1.7 < R_{\mathscr{S}} < 2$.

We briefly review some facts about degree theory. See [5], [13], [16], and [17] for further details. If G is a bounded open subset of $\mathbb{R}^n$ and $F: \bar{G} \to \mathbb{R}^n$ is a continuous map such that $F(x) \neq a$ for all $x \in \partial G$, then one can define an integer m, an algebraic count of the number of solutions in G of the equation $F(x) = a$. m is called the degree of F in G at a, $\deg(F, G, a)$. If F is C^1, a is a regular value of F and $\mathrm{L}_a = \{x \in G : F(x) = a\}$ then

$$\deg(F, G, a) = \sum_{x \in \mathrm{L}_a} \varepsilon(x),$$

where $\varepsilon(x)$ is the sign of the determinant of the Jacobian matrix of $F'(x)$. The degree has the following properties which will be used to analyze periodic points:

Normalization. If $F = I =$ the identity map and $a \notin \partial G$, then $\deg(I, G, a) = 1$ if $a \in G$ and $\deg(I, G, a) = 0$ if $a \notin G$.

Additivity. Suppose that G_1 and G_2 are bounded open subsets of $\mathbb{R}^n$, $G = G_1 \cup G_2$, and $F: G \to \mathbb{R}^n$ is a continuous map such that $F(x) \neq a$ for $x \in \partial G_1 \cup \partial G_2 \cup (G_1 \cap G_2)$. Then

$$\deg(F, G, a) = \deg(F, G_1, a) + \deg(F, G_2, a).$$

Homotopy. Suppose that G is a bounded open subset of $\mathbb{R}^n$ and that $F\colon \bar{G}\times[0,1]\to\mathbb{R}^n$ is a continuous map with $F_t(x)$ defined to be $F(x,t)$ for $x\in\bar{G}$ and $t\in[0,1]$. If $F(x,t)\neq a$ for all $(x,t)\in\partial G\times[0,1]$, then $\deg(F_t, G, a)$ is defined and constant for $0\leqslant t\leqslant 1$.

If G is an open subset of $\mathbb{R}^n$ (not necessarily bounded) and $F\colon G\to\mathbb{R}^n$ is a continuous map such that $\mathrm{L}_a=\{x\in G : F(x)=a\}$ is compact (possibly empty), then one can still define $\deg(F, G, a)$. Let H be any bounded open neighborhood of L_a with $\bar{H}\subset G$ and define $\deg(F,G,a)$ to be $\deg(F,H,a)$. Additivity of degree then shows that this definition is independent of the particular H chosen.

We will also need the *commutativity property* of degree theory. Suppose the U and V are open subsets of $\mathbb{R}^n$ and $f\colon U\to\mathbb{R}^n$ and $g\colon V\to\mathbb{R}^n$ are continuous maps. Assume that $Fix_{g\circ f}=\{x\in f^{-1}(V) :\ g(f(x))=x\}$ is compact (possibly empty). Then $Fix_{f\circ g}=\{y\in g^{-1}(U) : f(g(y))=y\}$ is homeomorphic to $Fix_{g\circ f}$ and

$$\deg(I-g\circ f, f^{-1}(V), 0)=\deg(I-f\circ g, g^{-1}(U), 0).$$

If G is an open subset of $\mathbb{R}^n$ and $f\colon G\to\mathbb{R}^n$ is a continuous map, let $f^{[j]}$ denote the composition of f with itself j times with its natural domain of definition in G. We call $x_0\in G$ a *periodic point of minimal period* p if $f^{[p]}(x_0)=x_0$ and $f^{[j]}(x_0)\neq x_0$ for $0\leqslant j<p$. Let $x_j=f^{[j]}(x_0)$ for $0\leqslant j<p$, and assume that there is $\varepsilon_j>0$ so that $f^{[p]}(y)\neq y$ for $0<\|y-x_j\|\leqslant\varepsilon_j$. Let $B_\varepsilon(\psi)=\{y \mid \|y-\psi\|<\varepsilon\}$. The commutativity property then implies that for $0\leqslant j<p$,

$$\deg(I-f^{[p]}, B_{\varepsilon_j}(x_j), 0)=\deg(I-f^{[p]}, B_{\varepsilon_0}(x_0), 0).$$

Here we will use degree theory to study holomorphic maps. One reference for the results needed is [19]. Suppose that G is a bounded open set in $\mathbb{C}^m$ and $F\colon G\to\mathbb{C}^m$ is holomorphic. We identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ using $(x_1+iy_1,\ x_2+iy_2, ...,\ x_m+iy_m)\leftrightarrow(x_1,\ y_1,\ x_2,\ y_2, ...,\ x_m, y_m)$. If $\Sigma=\{\zeta\in G : F(\zeta)=0\}$ is compact and nonempty, then Σ is a finite set. If $\zeta\in\Sigma$, $\varepsilon>0$ and $\overline{B_\varepsilon(\zeta)}\cap\Sigma=\{\zeta\}$, then $\deg(F, B_\varepsilon(\zeta), 0)$ is defined and $\deg(F, B_\varepsilon(\zeta), 0)\geqslant 1$. Thus the degree of F on G is bounded below by the number of elements of Σ. Also, if F_t is a homotopy of holomorphic maps avoiding $\partial B_\varepsilon(\zeta)$ (so $F_t^{-1}(0)\cap\partial B_\varepsilon(\zeta)=\{\zeta\}$ for all t) then $\deg(F_t, B_\varepsilon(\zeta), 0)$ is constant.

THEOREM 6.1. *Suppose m is a positive integer and $R>2$. Let $B_R=B_R(0,0)$. If $I\colon\mathbb{C}^2\to\mathbb{C}^2$ is the identity map, then the degree of $I-\Psi^{[m]}$ on B_R is defined, and* $\deg(I-\Psi^{[m]}, B_R, 0)=2^m$. *Therefore the equation $\Psi^{[m]}(w,z)=(w,z)$ has 2^m solutions (counting multiplicities) in B_R. These solutions are isolated, and* *there are at most* 2^m *distinct solutions. The map* Ψ *has infinitely many distinct periodic points.*

Proof. We use the results of Lemma 6.1 with $G = B_R$ and $F = I - \Psi^{[n]}$. The lemma asserts that $F_t(\zeta) \neq 0$ for $\zeta \in \mathbb{C}^2 \setminus G$, so $\Sigma_n = \{(w, z) \in G : \Psi^{[n]}(w, z) = (w, z)\}$ is a finite set and $\deg(F, B_R, 0)$ is defined. Lemma 6.1 also shows that if $F_t = I - \Psi_t^{[n]}$, then F_t is not zero on ∂G for $0 \leqslant t \leqslant 1$, so that $\deg(F_t, B_R, 0)$ is constant. When $t = 0$, this computes the number of roots (counting algebraic multiplicity) of $\Psi_0^{[n]}(w, z) = (w, z)$. But $\Psi_0^{[2k]}(w, z) = (w^{2^k}, z^{2^k})$ (n even, $n = 2k$) and $\Psi_0^{[2k-1]}(w, z) = (z^{2^{k-1}}, w^{2^k})$ (n odd, $n = 2k - 1$). Since $R > 1$ there 2^n roots, each with multiplicity 1. The remarks preceding the theorem then verify the conclusions about the number of solutions.

It remains to show that Ψ has infinitely many distinct periodic points. This will be proved using a result of Shub *et al.* [20]. An alternative proof with more precise assertions is given below. Suppose that $f\colon \mathbb{R}^q \to \mathbb{R}^q$ is a C^1 map and that $f(x_0) = x_0$. Assume that for every positive integer n, x_0 is an *isolated* fixed point of $f^{[n]}$: there is $\varepsilon_n(x_0) > 0$ so that $f^{[n]}(x) \neq x$ for $0 < \|x - x_0\| \leqslant \varepsilon_n(x_0)$. Shub and Sullivan show that there exists an integer N, independent of $n \geqslant 1$, such that $|\deg(I - f^{[n]}, B_{\varepsilon_n(x_0)}(x_0), 0)| \leqslant N$ for all $n \geqslant 1$.

We apply this result with $f = \Psi$. Suppose Ψ had only finitely many distinct periodic points, say $\zeta_1, \zeta_2, ..., \zeta_t$. Select $\varepsilon > 0$ so that $\overline{B_\varepsilon(\zeta_j)} \cap \overline{B_\varepsilon(\zeta_k)}$ is empty for all $j \neq k$. The Shub-Sullivan result then implies that there is an integer N with

$$0 \leqslant \deg(I - \Psi^{[n]}, B_\varepsilon(\zeta_j), 0) \leqslant N$$

for $n \geqslant 1$, $1 \leqslant j \leqslant t$. If R is large, the additivity property of degree gives

$$2^m = \deg(I - \Psi^{[m]}, B_R, 0) = \sum_{j=1}^{t} \deg(I - \Psi^{[m]}, B_\varepsilon(\zeta_j), 0) \leqslant tN$$

which is false for sufficiently large m. ∎

EXAMPLE 6.1. If $n = 1$, $\Psi(w, z) = (w, z)$ has one distinct fixed point, (0, 0), which has multiplicity 2 (consider Ψ_0 which has the same multiplicity at 0). If $n = 2$, $\Psi^{[2]}(w, z) = (w, z)$ has three distinct solutions: (0, 0), $\zeta_+ = (1 + i, 1 - i)$, and $\zeta_- = (1 - i, 1 + i)$. The solution (0, 0) has multiplicity 2, the other solutions have multiplicity 1, and $2 + 1 + 1 = 2^2$ as the theorem predicts. More information about the multiplicity of Ψ and its iterates at (0, 0), ζ_+, and ζ_- is provided in what follows.

We can be more precise about the existence of certain periodic points after computing some specific degrees.

THEOREM 6.2. *For each $n \geqslant 1$ select $\varepsilon_n > 0$ so that $\Psi^{[n]}(w, z) \neq (w, z)$ for $0 < \|(w, z)\| \leqslant \varepsilon_n$. Then for any ε with $0 < \varepsilon \leqslant \varepsilon_n$ and any $n \geqslant 1$,*

$$\deg(I - \Psi^{[n]}, B_\varepsilon(0, 0), 0) = 2.$$

Proof. We establish the theorem by a succession of homotopies inspired by the specific form of $\Psi^{[n]}$. We first claim that if $n \geqslant 2$,

$$\Psi^{[n]}(w, z) = (z + w^2 + (n-2)\, z^2 + P_n(w, z),\, z + w^2 + (n-1)\, z^2 + Q_n(w, z)),$$

where $P_n(w, z)$ and $Q_n(w, z)$ are polynomials in w and z and each term in each polynomial has degree $\geqslant 3$. This formula is easily established by induction.

Next consider the homotopy

$$\begin{aligned}(\Psi^{[n]})_s(w, z) = (&z + w^2 + (n-2)\, z^2 + sP_n(w, z),\\ &z + w^2 + (n-1)\, z^2 + sQ_n(w, z))\end{aligned}$$

for $0 \leqslant s \leqslant 1$. If $(\Psi^{[n]})_s(w, z) = (w, z)$ and $\|(w, z)\| = \varepsilon > 0$ for ε small, we see that $|z - w| = O(\varepsilon^2)$ so, for ε small enough, we can assume $|w| \geqslant \frac{1}{2}\varepsilon$ and $|z| \geqslant \frac{1}{2}\varepsilon$. $(\Psi^{[n]})_s(w, z) = (w, z)$ gives

$$w = z + w^2 + (n-2)\, z^2 + sP_n(w, z) \quad \text{and} \quad z = z + w^2 + (n-1)\, z^2 + sQ_n(w, z)$$

and therefore

$$\begin{aligned}0 &= (z + w^2 + (n-2)\, z^2 + sP_n(w, z))^2 + (n-1)\, z^2 + sQ_n(w, z)\\ &= nz^2 + R_n(w, z)\end{aligned}$$

where $R_n(w, z)$ is a polynomial all of whose terms have degree $\geqslant 3$. Therefore there is a constant M_n independent of $s \in [0, 1]$ such that $|R_n(w, z)| \leqslant M_n \varepsilon^3$ for $\|(w, z)\| \leqslant \varepsilon$. Since $|z| \geqslant \frac{1}{2}\varepsilon$, the equation $0 = nz^2 + R_n(w, z)$ is impossible for all sufficiently small ε. Thus there is $\varepsilon_n > 0$ so that $(\Psi^{[n]})_s(w, z) \neq (w, z)$ for $0 < \|(w, z)\| \leqslant \varepsilon_n$ and for ε with $0 < \varepsilon \leqslant \varepsilon_n$,

$$\deg(I - \Psi^{[n]}, B_\varepsilon, 0) = \deg(I - F_n, B_\varepsilon, 0)$$

if $F_n(w, z) = (z + w^2 + (n-2)\, z^2, z + w^2 + (n-1)\, z^2)$.

For fixed $n \geqslant 2$ we now consider the homotopy

$$H_\lambda(w, z) = (z + \lambda w^2 + \lambda(n-2)\, z^2, z + w^2 + \lambda(n-1)\, z^2)$$

with $0 \leqslant \lambda \leqslant 1$. We claim that if $0 < \|(w, z)\| \leqslant \varepsilon$ where $\varepsilon > 0$ is sufficiently small, then $H_\lambda(w, z) \neq (w, z)$ for $\lambda \in [0, 1]$. When $\lambda = 0$, $H_0 = \Psi$ and it is simple to check that $H_0(w, z) = (w, z)$ if and only if $(w, z) = (0, 0)$. So assume $0 < \lambda \leqslant 1$. Then $H_\lambda(w, z) = (w, z)$ is exactly

$$w = z + \lambda w^2 + \lambda(n-2)\, z^2 \qquad \text{and} \qquad z = z + w^2 + \lambda(n-1)\, z^2.$$

The second of these equations shows that $w = \pm iz\sqrt{\lambda(n-1)}$ so $w = 0$ exactly when $z = 0$. If we assume $z \neq 0$ then substitution in the first equation yields

$$\pm iz\sqrt{\lambda(n-1)} = z - \lambda^2(n-1)\, z^2 + \lambda(n-2)\, z^2$$

so that

$$z = \frac{-1 \pm i\sqrt{\lambda(n-1)}}{\lambda(n-2) - \lambda^2(n-1)}.$$

In order for z to be a non-zero solution, $\lambda(n-2) - \lambda^2(n-1)$ must not be 0. Then there is $c_n > 0$ so that

$$|z| \geqslant \frac{\sqrt{1 + \lambda(n-1)}}{\lambda\, |(n-2) - \lambda(n-1)|} \geqslant c_n \qquad \text{for} \quad 0 \leqslant \lambda \leqslant 1.$$

The homotopy H_λ can be used to compute degree for sufficiently small $\varepsilon > 0$.

So there is $\tilde{\varepsilon}_n > 0$ so that when $0 < \varepsilon \leqslant \tilde{\varepsilon}_n$,

$$\deg(I - \Psi^n, B_\varepsilon, 0) = \deg(I - \Psi, B_\varepsilon, 0).$$

Example 6.1 shows that the degree on the right is 2. ∎

$\zeta_+ = (1+i, 1-i)$ and $\zeta_- = (1-i, 1+i)$ were defined in Example 6.1. They must also be fixed points of $\Psi^{[2m]}$ for any positive integer m.

THEOREM 6.3. *Select* $\varepsilon_m > 0$ *so that* $(I - \Psi^{[2m]})(w, z) \neq 0$ *for* $0 < \|(w, z) - \zeta_s\| \leqslant \varepsilon_m$ *if* s *is either* $+$ *or* $-$. *If* $0 < \varepsilon \leqslant \varepsilon_m$, *then* $\deg(I - \Psi^{[2m]}, B_\varepsilon(\zeta_s), 0) = 1$.

Proof. The complex Fréchet derivative of $\Phi = \Psi^{[2]}$ at (w, z) is given by the 2×2 matrix

$$\Psi'(w, z) = \begin{pmatrix} 2w & 1 \\ 2w & 1 + 2z \end{pmatrix}.$$

$\Psi'(w, z)$ acts on $\mathbb{C}^2$ by multiplying 2×1 column vectors on the left.

If $\Phi(w, z) = (w, z)$ and $I - (\Phi'(w, z))^m$ is invertible, then $\deg(I - \Phi^m, B_\varepsilon, 0) = 1$ for all sufficiently small $\varepsilon > 0$. Here $(w, z) = \zeta_s$. If we can prove that every eigenvalue λ of $\Phi'(\zeta_s)$ satisfies $|\lambda| > 1$, then $I - \Phi'(\zeta_s)$ will be invertible. Since μ is an eigenvalue of $\Phi'(\zeta_s)^m$ if and only if $\mu = \lambda^m$ for some λ which is an eigenvalue of $\Phi'(\zeta_s)$, every eigenvalue of $\Phi'(\zeta_s)^m$ will satisfy $|\mu| > 1$. This implies that $I - \Phi'(\zeta_s)^m$ is invertible, and the theorem will be proved.

We complete the proof by computing the eigenvalues of $\Phi'(\zeta_s)$. $\det \Phi'(w, z) = 4wz$ and $\operatorname{tr} \Phi'(w, z) = 1 + 2(w + z)$. Therefore when $(w, z) = \zeta_s$, $4wz = 8$ and $1 + 2(w + z) = 5$, so the eigenvalues are the roots of $\lambda^2 - 5\lambda + 8 = 0$. These roots are $\lambda = (5 \pm \sqrt{7}\, i)/2$, which satisfy $|\lambda| > 1$. ∎

Theorem 6.4. *If $p \geqslant 2$ is a prime, Ψ and $\Phi = \Psi^{[2]}$ must both have periodic points of minimal period p.*

Proof. Suppose $p > 2$. For $R > 2$, Theorem 6.1 asserts that $\deg(I - \Phi^{[p]}, B_R, 0) = 2^{2p}$. We again apply additivity of degree to underestimate this by $\deg(I - \Phi^{[p]}, B_\varepsilon, 0) + \deg(I - \Phi^{[p]}, B_\varepsilon(\zeta_+), 0) + \deg(I - \Phi^{[p]}, B_\varepsilon(\zeta_-), 0)$. For sufficiently small ε the previous results show that this sum is 4. Therefore the map $\Phi^{[p]}$ must have additional fixed points which are not fixed points of Φ. But since p is prime, any fixed point ζ of $\Phi^{[p]}$ which is not a fixed point of Φ must satisfy $\Phi^{[j]}(\zeta) \neq \zeta$ for $1 \leqslant j < p$. Such ζ's are fixed points of minimal period p for Φ.

Only $(0, 0)$ is a fixed point for Ψ and its multiplicity is 2. The degree inequality above becomes $2^p \geqslant 2$ for $p > 2$, thus showing the existence of periodic points of minimal period p for Ψ as well.

We complete the proof of the theorem by verifying the case $p = 2$. Each ζ_s is a periodic point of minimal period 2 for Ψ. $\Phi^{[2]}$ has degree $2^4 = 16$ on a large ball, and the fixed points of Φ are 0 and the two ζ_s's, which contribute only 4 to that count. Thus (as above) there are additional fixed points of $\Phi^{[2]}$ which cannot be fixed points of Ψ. ∎

We can completely describe the periodic points of $\Psi^{[n]}$ for $n = 3$ and $n = 4$. We describe the easier case when $n = 3$ completely and summarize what happens when $n = 4$.

Complex conjugation is the real linear involution of $\mathbb{C}^2$ defined by $\mathscr{C}(w, z) = (\bar{w}, \bar{z})$. Of course, the QF recurrence (the map Ψ) and $\mathscr{C}$ commute which helps further analysis of the periodic points.

Lemma 6.2. *Suppose $\zeta = (w, z) \in \mathbb{C}^2$ and $\zeta \neq (0, 0)$. If $\Psi^{[3]}(\zeta) = \zeta$, then $\mathscr{C}(\zeta) \notin \{\zeta, \Psi(\zeta), \Psi^{[2]}(\zeta)\}$.*

Proof. $(0, 0)$ is the only periodic point in $\mathbb{R}^2$ of Ψ. But if $\mathscr{C}(\zeta) = \zeta$, then $\zeta \in \mathbb{R}^2$ which contradicts the assumption that $\zeta \neq (0, 0)$. Therefore $\mathscr{C}(\zeta) \neq \zeta$.

If $\mathscr{C}(\zeta) = \Psi(\zeta)$, then $(\bar{w}, \bar{z}) = (z, z + w^2)$. Then $w = \bar{w} + w^2$, so if $w = x + iy$ the equation

$$x + iy = x - iy + (x^2 - y^2) + 2ixy$$

results. Taking imaginary parts of both sides yields the equation $y(1 - x) = 0$ while taking real parts gives $x^2 - y^2 = 0$. So either $y = 0$ or $x = 1$. If $y = 0$, the second equation shows that $x = 0$ so $w = 0$ and then ζ must be $(0, 0)$, which is again a contradiction. If $x = 1$, then $y = \pm 1$. Then $w = 1 \pm i$ so $z = 1 \mp i$, and it is easy to see that $\Psi^{[3]}(1 \pm i, 1 \mp i) \neq (1 \pm i, 1 \mp i)$.

Finally, suppose that $\mathscr{C}(\zeta) = \Psi^{[2]}(\zeta)$. Since $\mathscr{C}$ and Ψ commute, we see that $\mathscr{C}(\Psi(\zeta)) = \Psi^{[3]}(\zeta) = \zeta$ so that $\Psi(\zeta) = \mathscr{C}(\zeta)$ since $\mathscr{C}$ is an involution. But we previously showed that this equation has no solutions under the hypotheses of this lemma. ∎

Let $T = \{\eta \in \mathbb{C}^2 : \Psi^{[3]}(\eta) = \eta\}$. T is a finite set whose points all have distance at most 2 from the origin. There is $\varepsilon > 0$ so that $\Psi^{[3]}(v) \neq v$ when $0 < \|v - \eta\| < \varepsilon$ and $\eta \in T$. Note that $\deg(I - \Psi^{[3]}, B_\varepsilon(\eta), 0) \geqslant 1$ and $\deg(I - \Psi^{[3]}, B_R(0), 0) = 2^3 = 8$ for any $R > 4$. Previous results show that there must be some ζ of minimal period 3. The lemma then asserts that ζ, $\Psi(\zeta)$, $\Psi^{[2]}(\zeta)$, $\mathscr{C}(\zeta)$, $\mathscr{C}(\Psi(\zeta))$, and $\mathscr{C}(\Psi^{[2]}(\zeta))$ must all be distinct. These 6 points each contribute at least 1 to the total degree count. Since $\deg(I - \Psi^{[3]}, B_\varepsilon(0), 0) = 2$ we have accounted for all elements of T, and have verified almost all of the following.

THEOREM 6.5. *Ψ has precisely two distinct periodic orbits of minimal period 3. If one orbit is $\{\zeta, \Psi(\zeta), \Psi^{[2]}(\zeta)\}$, the other orbit is $\{\mathscr{C}(\zeta), \Psi(\mathscr{C}(\zeta)), \Psi^{[2]}(\mathscr{C}(\zeta))\}$. If η is any periodic point of Ψ of minimal period 3, then $I - (\Psi^{[3]})'(\eta)$ is nonsingular.*

Proof. The final assertion of the theorem is a consequence of an additional result of [19]: suppose that $\zeta \in \mathbb{C}^n$, $\varepsilon > 0$, $F(\zeta) = a$, and $F: B_\varepsilon(\zeta) \to \mathbb{C}^n$ is holomorphic. If $\deg(F, B_\varepsilon(\zeta), a)$ is defined and equal to 1, then $F'(\zeta)$ is nonsingular. Take $F = I - \Psi^{[3]}$ here, proving the final statement of the theorem. ∎

Computer-assisted computation reveals that the minimal period 3 points of Ψ are of the form $(w, -\frac{1}{2}w - \frac{1}{2}w^2 + \frac{1}{4}w^3 + \frac{1}{4}w^5)$ where w is any root of $w^6 - 3w^2 + 6 = 0$. Approximate numerical values are $w = \pm a \pm bi$ or $\pm ci$ with $a \approx 1.1776\ 50699$ and $b \approx 0.4573\ 953100$ and $c \approx 1.5346\ 99123$.

The analysis of the fixed point set of $\Psi^{[4]}$ is considerably more complicated. The total degree count is $2^4 = 16$. The origin accounts for 2 of this total, and each of ζ_+ and ζ_- for 1. None of these is a minimal period 4 point. The interaction between $\mathscr{C}$ and $\Psi^{[n]}$ and its consequences for fixed point sets have been investigated. The proofs use rather specific homotopies. The arguments are complicated but similar to some of the proofs given previously. They will not be given here. Some of these results may seem familiar, such as the following:

LEMMA 6.3. *For any* $R>2$, $\deg(I-\mathscr{C}\Psi^{[n]}, B_R(0), 0)$ *is defined and equal to* 2^n.

Of course, $\mathscr{C}$ is not holomorphic, and this has consequences which may not be anticipated and which emphasize that degree counts the solutions of an equation algebraically. If $\mathscr{C}\Psi^{[n]}(\zeta)=\zeta$, then $\Psi^{[2n]}(\zeta)=\zeta$. Therefore the solutions of $\mathscr{C}\Psi^{[n]}(\zeta)=\zeta$ are isolated, and there can be no more than 4^n-1 distinct such solutions (the origin is counted twice in the degree of $\Psi^{[2n]}$).

LEMMA 6.4. *For each* $n\geqslant 1$, *select* $\varepsilon_n>0$ *so that* $\mathscr{C}\Psi^{[n]}(\zeta)\neq\zeta$ *for* $0<\|\zeta\|<\varepsilon_n$. *Then for any* $\varepsilon>0$ *with* $0<\varepsilon<\varepsilon_n$, $\deg(I-\mathscr{C}\Psi^{[n]}, B_\varepsilon(0), 0)=0$.

The idea of the proof is to use a homotopy and obtain the equality

$$\deg(I-\mathscr{C}\Psi^{[n]}, B_\varepsilon(0), 0)=\deg(I-\mathscr{C}\Psi, B_\varepsilon(0), 0).$$

The details of this argument are intricate but the final step of this lemma's proof uses a technique not previously employed here. The final step follows.

Proof (of the lemma for $n=1$). We write $w=x_1+ix_2$ and $z=x_3+ix_4$. Then

$$\mathscr{C}\Psi(w,z)=(x_3-ix_4,\, x_3+(x_1^2-x_2^2)-i(x_4+2x_1x_2))$$

and we think of $\mathscr{C}\Psi$ as a map from $\mathbb{R}^4$ to $\mathbb{R}^4$. Then

$$(I-\mathscr{C}\Psi)(x_1,x_2,x_3,x_4)=(x_1-x_3,\, x_2+x_4,\, x_2^2-x_1^2,\, 2x_4+2x_1x_2).$$

If $a>0$, we claim that $(0,0,-a^2,0)$ is a regular value for $I-\mathscr{C}\Psi$ on $B_\varepsilon(0)$, and that $(I-\mathscr{C}\Psi)^{-1}(0,0,-a^2,0)$ consists of the two points $\eta_+=(a,0,a,0)$ and $\eta_-=(-a,0,-a,0)$ for a sufficiently small. We verify this claim.

Certainly $(I-\mathscr{C}\Psi)(\eta_\pm)=(0,0,-a^2,0)$. Conversely, if $(I-\mathscr{C}\Psi)(x_1, x_2, x_3, x_4)=(0,0,-a^2,0)$, then

$$x_1-x_3=0, \qquad x_2+x_4=0, \qquad x_2^2-x_1^2=-a^2, \qquad 2x_4+2x_1x_2=0$$

and therefore $x_1=x_3$, $x_2=-x_4$, and $2x_4+2x_1x_2=2x_4(1-x_1)=0$. If $x_4\neq 0$, then $x_1=1$ which is not possible for $\varepsilon<1$. Thus $x_4=0$ and $x_2=-x_4=0$, so that $-x_1^2=-a^2$ and $x_1=\pm a$. Further computation gives

$$(I-\mathscr{C}\Psi)'(x_1,x_2,x_3,x_4)=\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -2x_1 & 2x_2 & 0 & 0 \\ 2x_2 & 2x_1 & 0 & 2 \end{pmatrix}$$

which has determinant $-4x_1(1-x_1)+4x_2^2$. At η_+, this is $-4a(1-a)$, and at η_-, it is $4a(1+a)$. When $0<a<1$, both determinants are non-zero, establishing the regularity asserted. The determinants have opposite signs, so the algebraic count of the degree is indeed 0. ∎

The last two results imply that $\mathscr{C}\Psi^{[2]}(\zeta)=\zeta$ has non-zero solutions.

LEMMA 6.5. *There is $\zeta \neq (0,0)$ such that $\mathscr{C}\Psi^{[2]}(\zeta)=\zeta$.*

Proof. For R large, $\deg(I-\mathscr{C}\Psi^{[n]}, B_R(0), 0)=2^n$, so $\deg(I-\mathscr{C}\Psi^{[2]}, B_R(0), 0)=4$. For sufficiently small $\varepsilon>0$, $\deg(I-\mathscr{C}\Psi^{[2]}, B_\varepsilon(0), 0)=0$. If $U=B_R(0)\backslash\overline{B_\varepsilon(0)}$, additivity of degree implies that $\deg(I-\mathscr{C}\Psi^{[2]}, U, 0)=4$. Therefore there is $\zeta\in U$ with $\mathscr{C}\Psi^{[2]}(\zeta)=\zeta$. ∎

An exact description of the orbit structure of the fixed point set of $\Psi^{[4]}$ requires a great deal of specific computation differing in detail but not in nature from what has already been written above. We will not show these details. We state the final result, however.

THEOREM 6.6. *Ψ has precisely three distinct periodic orbits of minimal period 4. If $\Psi^{[4]}(\zeta)=\zeta$ for $\zeta\in\mathbb{C}^2$ and $\zeta\neq 0$, then $I-(\Psi^{[4]})'(\zeta)$ is non-singular. There are precisely four distinct solutions of $\mathscr{C}\Psi^{[2]}(\zeta)=\zeta$ and these solutions make up one of the minimal orbits of period 4, an orbit which is conjugate to itself. The other two orbits are complex conjugates of each other.*

The fixed points of minimal period 4 can also be described as the collection of points $(w, P(w))$ in $\mathbb{C}^2$ where P is a specific polynomial of degree 11, and the w's are all roots of another specific polynomial of degree 12. Both polynomials have real rational coefficients.

7. ASYMPTOTICALLY PERIODIC DOUBLY INFINITE COMPLEX SEQUENCES

In this section we continue our use of complex numbers in the QF recurrence. Any initial conditions other than $(0,0)$ then will be part of uncountably many distinct doubly infinite sequences satisfying the QF recurrence. But we can, in fact, specify the behavior of such a sequence as $n\to-\infty$ rather strictly. If q is a complex number, write $\sqrt{q}$ to indicate the principal branch of square root, defined here by restricting its argument to the interval $(-\frac{\pi}{2}, \frac{\pi}{2}]$. We further define the open quadrants of the complex plane: if j is one of the integers $\{1,2,3,4\}$, then $Q_j=\{z\in\mathbb{C}:(j-1)\frac{\pi}{2}<\operatorname{Arg} z<j(\frac{\pi}{2})\}$ (here $\operatorname{Arg} z\in[0,2\pi)$). Let $\mathscr{D}_+$ (respectively, $\mathscr{D}_-$) denote $Q_1\times Q_4$ (respectively, $Q_4\times Q_1$). Both are subsets of $\mathbb{C}^2$, and $\zeta_\pm\in\mathscr{D}_\pm$. One

realization of $\Phi^{[-1]}$ is the mapping $(w, z) \mapsto (\sqrt{w - \sqrt{z-w}}, \sqrt{z-w})$ which we call Θ.

LEMMA 7.1. *Θ is a holomorphic mapping from $\mathscr{D}_+$ (respectively, $\mathscr{D}_-$) to itself. The only fixed point of Θ in $\mathscr{D}_+$ (respectively, $\mathscr{D}_-$) is ζ_+ (respectively, ζ_-).*

Proof. We verify the + variant, noting that − is similar. If $(w, z) \in D_+ = Q_1 \times Q_4$, then $-w \in Q_3$ so $z - w$ has argument in $(\pi, 2\pi)$. Therefore $\sqrt{z-w} \in Q_4$. $-\sqrt{z-w}$ must be in Q_2 so $w - \sqrt{z-w}$ has argument in $(0, \pi)$, so that $\sqrt{w - \sqrt{z-w}} \in Q_1$. Therefore $\Theta(\mathscr{D}_+) \subseteq \mathscr{D}_+$. The fixed points of Θ are as indicated (see Example 19). ∎

Both $\mathscr{D}_+$ and $\mathscr{D}_-$ are open subsets of $\mathbb{C}^2$ and hyperbolic complex manifolds, and Θ is a holomorphic self-mapping of each domain with a unique fixed point. This fixed point is attractive (the eigenvalues of the inverse mapping computed in Theorem 6.3 all have modulus greater than 1). General results on mappings of hyperbolic manifolds (the generalized Schwarz lemma) show that given any $(z, w) \in \mathscr{D}_+$ (respectively, $\mathscr{D}_-$), the iterates $\Theta^{[n]}(w, z)$ all have limit ζ_+ (respectively, ζ_-) as $n \to \infty$. See [8] or [10]. We describe the idea of the proof for $\mathscr{D}_+$. Similar reasoning holds for $\mathscr{D}_-$. There exists a metric ρ on $\mathscr{D}_+$ which gives the usual topology on $\mathscr{D}_+$, makes $(\mathscr{D}_+, \rho)$ a complete metric space, and satisfies $\rho(\Theta(\zeta_1), \Theta(\zeta_2)) \leqslant \rho(\zeta_1, \zeta_2)$ for all $\zeta_1, \zeta_2 \in \mathscr{D}_+$. Combining this with the fact that Θ has an attractive fixed point $\zeta_+ \in \mathscr{D}_+$, one can prove that $\lim_{k \to \infty} \Theta^{[k]}(\zeta) = \zeta_+$ for all $\zeta \in \mathscr{D}_+$. We have almost proved the following result:

THEOREM 7.1. *Suppose $(w, z) \in \mathbb{C}^2$ and $(w, z) \neq (0, 0)$. Then there is a sequence $\{z_n\}_{n \in \mathbb{Z}}$ which satisfies the QF recurrence so that either $z_{2n} \to 1 + i$ and $z_{2n+1} \to 1 - i$ as $n \to -\infty$ or $z_{2n} \to 1 - i$ and $z_{2n+1} \to 1 + i$ as $n \to -\infty$.*

Proof. Suppose $(w, z) \neq (0, 0)$ is given. The recurrence produces z_n for $n > 1$. The preceding lemma and the remarks about hyperbolicity show that if we can find some "ancestor" of (w, z) for the QF recurrence which lies in either $\mathscr{D}_+ = Q_1 \times Q_4$ or $\mathscr{D}_- = Q_4 \times Q_1$ we are done. Thus we need to show that $\bigcup_{k=1}^{\infty} \Psi^{[k]}(\mathscr{D}_+ \cup \mathscr{D}_-) \supset \mathbb{C}^2 \backslash \{(0, 0)\}$. We will in fact show that $\Psi^{[5]}(\mathscr{D}_+ \cup \mathscr{D}_-) \supset \mathbb{C}^2 \backslash \{(0, 0)\}$. Here $\mathbb{R}_{\geqslant 0}$ will denote the non-negative reals.

STEP 1. $\Psi(\mathscr{D}_+ \cup \mathscr{D}_-) \supset G_1 = \{(a, b) \in \mathbb{C}^2 : a \in Q_4, \operatorname{Im} b \geqslant 0\} \cup \{(a, b) \in \mathbb{C}^2 : a \in Q_1, \operatorname{Im} b \leqslant 0\}$.

Proof of Step 1. Suppose that $a \in Q_4$ and $\operatorname{Im} b \geqslant 0$. If $\Psi(w, z) = (z, z + w^2) = (a, b)$, then $z = a \in Q_4$ and $w^2 = b - a$. Since $\operatorname{Im} b \geqslant 0$ and $\operatorname{Im}(-a) > 0$, then $\operatorname{Im}(b - a) > 0$ and there is $w \in Q_1$ with $w^2 = b - a$. A similar argument shows that $\Psi(\mathscr{D}_-) \supset \{(a, b) : a \in Q_1, \operatorname{Im} b \leqslant 0\}$.

STEP 2. $\Psi(G_1) \supset G_2 = \{(a, b) \in \mathbb{C}^2 : \operatorname{Im} a \geqslant 0, \operatorname{Im} b \leqslant 0, \operatorname{Im} a - \operatorname{Im} b > 0\} \cup \{(a, b) \in \mathbb{C}^2 : \operatorname{Im} a \leqslant 0, \operatorname{Im} b \geqslant 0, \operatorname{Im} b - \operatorname{Im} a > 0\} \supset G_1$.

Proof of Step 2. $G_1 \supset \{(w, z) \colon w \in Q_4, \operatorname{Im} z \geqslant 0\}$. If $\operatorname{Im} a \geqslant 0$, $\operatorname{Im} b \leqslant 0$, and $\operatorname{Im} a - \operatorname{Im} b > 0$, we want $\Psi(w, z) = (z, z + w^2) = (a, b)$. Choose $z = a$ so $\operatorname{Im} z \geqslant 0$. We want $w^2 + z = b$ for $w \in Q_4$, so w^2 must be $b - a$. Since $\operatorname{Im}(a - b) = \operatorname{Im} a - \operatorname{Im} b > 0$, $\operatorname{Im}(b - a) < 0$, and there must be $w \in Q_4$ with $w^2 = b - a$.

If $\operatorname{Im} a \leqslant 0$, $\operatorname{Im} b \geqslant 0$, and $\operatorname{Im} b - \operatorname{Im} a > 0$, we want (w, z) with $w \in Q_1$ and $\operatorname{Im} z \leqslant 0$ so that

$$\Psi(w, z) = (z, z + w^2) = (a, b).$$

Thus $z = a$ and w^2 must be $b - a$. Here $\operatorname{Im}(b - a) > 0$, so there is $w \in Q_1$ with $w^2 = b - a$.

That $G_2 \supset G_1$ is clear.

STEP 3. $\Psi(G_2) \supset G_3 = \{(a, b) \in \mathbb{C}^2 : \operatorname{Im} a \neq 0 \textit{ or } \operatorname{Im} a = 0 \textit{ and } b - a \notin \mathbb{R}_{\geqslant 0}\} \supset G_2$.

Proof of Step 3. $G_1 \subset G_2$ so $\Psi(G_1) \subset \Psi(G_2)$ and $G_2 \subset \Psi(G_2)$. If $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and $\operatorname{Im} a > 0$, take $z = a$ and select $w \in \mathbb{C}$ such that $\operatorname{Im} w \leqslant 0$ and $w^2 = b - z = b - a$. We can always select such a w. Then $(w, z) \in G_2$ and $\Psi(w, z) = (a, b)$.

If $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and $\operatorname{Im} a < 0$, take $z = a$ and select $w \in \mathbb{C}$ such that $\operatorname{Im} w \geqslant 0$ and $w^2 = b - z = b - a$. We can always select such a w. Then $(w, z) \in G_2$ and $\Psi(w, z) = (a, b)$.

Finally, suppose $a \in \mathbb{R}$ and $b - a \notin \mathbb{R}_{\geqslant 0}$. Let $z = a$. Then there is $w \in \mathbb{C}$ with $\operatorname{Im} w < 0$ and $w^2 = b - z = b - a$, so $(w, z) \in G_2$ and $\Psi(w, z) = (a, b)$.

Again, that $G_3 \supset G_2$ is also clear.

STEP 4. $\Psi(G_3) \supset G_4 = \{\mathbb{C}^2 \setminus \{(0, 0)\}\} \setminus \{(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\} : b \geqslant a \textit{ and } a \geqslant \sqrt{b - a}\} \supset G_3$.

Proof of Step 4. Surely $\Psi(G_3) \supset \Psi(G_2) \supset G_3$. Now suppose $(a, b) \in \mathbb{R}^2$ and $b - a \in \mathbb{R}_{\geqslant 0}$. We want to discover when there is $(w, z) \in G_3$ with $\Psi(w, z) = (z, z + w^2) = (a, b)$. Then $z = a$ and $w^2 = b - a \geqslant 0$ so $w = \pm\sqrt{b - a} \in \mathbb{R}$. The condition for membership in G_3 translates to $a - \sqrt{b - a} \notin \mathbb{R}_{\geqslant 0}$. Thus there is no suitable element of G_3 when $a \geqslant \sqrt{b - a}$.

STEP 5. $\Psi(G_4) \supset G_5 = \mathbb{C}^2 \setminus \{(0, 0)\}$.

Proof of Step 5. $\Psi(G_4) \supset \Psi(G_3) \supset G_4$ from the previous step, so we need only discover why (a, b) with real a and b not both 0 satisfying $b \geqslant a$ and $a \geqslant \sqrt{b-a}$ are in Ψ's image of G_4. Since $\Psi(w, z) = (z, z + w^2) = (a, b)$, again z must equal a and $w = \pm\sqrt{b-a}$. For (w, z) to be an element of G_4, either $w > z = a$ or $z \geqslant w$ and $w < \sqrt{z-w}$. The first alternative, $w > a$, cannot occur, since we know that $|w| = \sqrt{b-a} \leqslant a$. But this implies $z \geqslant w$. We just must check $w < \sqrt{z-w}$, but this translates to $\pm\sqrt{b-a} < \sqrt{a-(\pm\sqrt{b-a})}$. Take $w = -\sqrt{b-a}$. Then we need $-\sqrt{b-a} < \sqrt{a+\sqrt{b-a}}$. This can only fail if $a = b = 0$ which is impossible, since not both can be 0.

We have shown that $\Psi^{[j]}(\mathscr{D}_+ \cup \mathscr{D}_-) \supset G_j$ with $G_5 = \mathbb{C}^2 \setminus \{(0,0)\}$, and so have shown that every element of $\mathbb{C}^2 \setminus \{(0,0)\}$ has some ancestor in $\mathscr{D}_+ \cup \mathscr{D}_-$ as required. ∎

A proof of the previous result "looking backwards"—getting inverses of Ψ applied to $(w, z) \in \mathbb{C}^2 \setminus \{(0,0)\}$ to lie in $\mathscr{D}_+ \cup \mathscr{D}_-$—can be written. The proof given above seems more natural. Examples show that there is no uniqueness of the sequence $\{z_n\}$ or of the parity of n for which $z_n \to 1 + i$.

Numerical and graphical exploration of the results of choosing "the other" square root in the definition of Θ (that is, $-\sqrt{\ }$) indicates interesting behavior in the sequence $\{z_n\}$ as $n \to -\infty$. Define $\mathscr{S}_+$ and $\mathscr{S}_-$ by $\mathscr{S}_+(w, z) = (\sqrt{z-w}, w)$ and $\mathscr{S}_-(w, z) = (-\sqrt{z-w}, w)$, respectively, so that $\Theta = \mathscr{S}_+ \circ \mathscr{S}_+$. Some compositions of $\mathscr{S}_+$ and $\mathscr{S}_-$ correspond to orbits of powers of Ψ and additional results about asymptotically periodic sequences can be proved.

For example, the conjugate-invariant minimal period 4 orbit of Ψ (see Theorem 6.6) is $A \xrightarrow{\Psi} B \xrightarrow{\Psi} C \xrightarrow{\Psi} D \xrightarrow{\Psi} A$ where the points A, B, C, and D in $\mathbb{C}^2$ are

$$A \approx (1.5148 - 0.39460i,\ -0.62408 + 1.5901i);$$

$$B \approx (-0.62408 + 1.5901i,\ 1.5148 + 0.39460i);$$

$$C \approx (1.5148 + 0.39460i,\ -0.62408 - 1.5901i);$$

$$D \approx (-0.62408 - 1.5901i,\ 1.5148 - 0.39460i).$$

Then $A \xrightarrow{\mathscr{S}_-} D \xrightarrow{\mathscr{S}_+} C \xrightarrow{\mathscr{S}_-} B \xrightarrow{\mathscr{S}_+} A$ and the mapping $\mathscr{S}_- \circ \mathscr{S}_+ \circ \mathscr{S}_- \circ \mathscr{S}_+$ has A as an attractive fixed point. A result similar to Theorem 7.1 about asymptotically periodic sequences of period 4 can be proved, so that as $n \to -\infty$, generically sequences would approach the repeated pattern $1.5148 - 0.39460i$, $-0.62408 + 1.5901i$, $1.5148 + 0.39460i$, $-0.62408 - 1.5901i$.

The other two minimal period 4 orbits of Ψ are conjugates of each other. If $\mathscr{S}_+ \circ \mathscr{S}_- \circ \mathscr{S}_- \circ \mathscr{S}_-$ is used as the backwards mapping, the limiting pattern becomes $0.60196+0.69713i$, $-0.95125-0.54455i$, $-1.0749+0.29474i$, $-0.46654+1.3307i$, while if $\mathscr{S}_- \circ \mathscr{S}_- \circ \mathscr{S}_- \circ \mathscr{S}_+$ is used, the limiting pattern becomes $0.60196-0.69713i$, $-0.95125+0.54455i$, $-1.0749-0.29474i$, $-0.46654-1.3307i$.

$\Psi^{[4]}$'s orbits and the resulting patterns are perhaps exceptionally simple. For example, experiments suggest more complicated relationships between the orbits of $\Psi^{[3]}$ and limiting behavior of tails of sequences satisfying the QF recurrence.

Compositions of $\mathscr{S}_-$ alone give results which seem to depend strongly on the initial conditions. For certain initial conditions, there may be sets of attractive points in the left half-plane. But qualitatively new phenomena also occur. Some sequences seem to approach smooth embedded closed curves in $\mathbb{C}$ as $n \to -\infty$, while other sequences seem to approach fractal sets.

We certainly do not understand now the set of all ancestors of general initial conditions in $\mathbb{C}^2$ very well.

8. ARGUMENT INCREASING DOUBLY INFINITE COMPLEX SEQUENCES

We discuss in detail a class of solutions to the QF recurrence. Understanding the behavior of these solutions will allow us to improve our knowledge of the qualitative properties of the function L defined earlier. We first label several subsets of $\mathbb{C}$:

$$\mathscr{H}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \qquad \text{the upper half-plane;}$$

$$\mathscr{H}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}, \qquad \text{the lower half-plane;}$$

$$\mathbb{R}_{>0} = \{z \in \mathbb{R} : z > 0\}, \qquad \text{the right open half-line;}$$

$$\mathbb{R}_{\geqslant 0} = \{z \in \mathbb{R} : z \geqslant 0\}, \qquad \text{the right closed half-line.}$$

We will need to be quite careful about our use of argument of a complex number. In what follows, if $z \neq 0$ is a complex number, $\operatorname{Arg} z$ will be the unique number in $[0, 2\pi)$ so that $z = |z|\, e^{i \operatorname{Arg} z}$, and $\arg z$ will be the unique number in $(-\pi, \pi]$ so that $z = |z|\, e^{i \arg z}$. These arguments coincide in $\overline{\mathscr{H}_+} \setminus \{0\}$.

The proof of the lemma below is left to the reader:

LEMMA 8.1. *If* $w, z \in \mathscr{H}_+ \cup \mathbb{R}_{>0}$ *and* $0 \leqslant \operatorname{Arg} w \leqslant \operatorname{Arg} z < \pi$, *then* $z + w^2 \neq 0$ *and*

$$\min(\operatorname{Arg} z, \operatorname{Arg}(w^2)) \leqslant \operatorname{Arg}(z + w^2) \leqslant \max(\operatorname{Arg} z, \operatorname{Arg}(w^2)).$$

If $\operatorname{Arg} z \neq \operatorname{Arg}(w^2)$, *then strict inequality holds in both relations above.*

Simple examples show that if $0 \leqslant \operatorname{Arg} z < \operatorname{Arg} w < \pi$, the conclusions of Lemma 8.1 may not hold.

LEMMA 8.2. *Suppose that* $z_0, z_1 \in \mathscr{H}_+ \cup \mathbb{R}_{>0}$ *and* $\operatorname{Arg} z_0 \leqslant \operatorname{Arg} z_1 \leqslant 2 \operatorname{Arg} z_0$. *For* $j > 1$ *define* $z_j = z_{j-1} + z_{j-2}^2$ *and assume that* $z_j \in \mathscr{H}_+ \cup \mathbb{R}_{\geqslant 0}$ *for* $0 \leqslant j \leqslant N$ *with* $N \geqslant 1$. *Then* $z_j \neq 0$ *for* $0 \leqslant j \leqslant N+1$ *and*

$$\operatorname{Arg} z_{j-1} \leqslant \operatorname{Arg} z_j \leqslant 2 \operatorname{Arg} z_{j-1}$$

for $1 \leqslant j \leqslant N+1$. *If* $2 \operatorname{Arg} z_0 > \operatorname{Arg} z_1$, *these inequalities are strict for* $2 \leqslant j \leqslant N+1$.

Proof. Since $\operatorname{Arg} z_0 \leqslant \operatorname{Arg} z_1 \leqslant 2 \operatorname{Arg} z_0 = \operatorname{Arg} z_0^2$, Lemma 8.1 implies that $z_2 \neq 0$. If further $\operatorname{Arg} z_1 < \operatorname{Arg}(z_0^2)$, then Lemma 8.1 declares that $\operatorname{Arg} z_2 = \operatorname{Arg}(z_1 + z_0^2) > \min(\operatorname{Arg} z_1, \operatorname{Arg}(z_0^2)) = \operatorname{Arg} z_1$ and $\operatorname{Arg} z_2 < \max(\operatorname{Arg} z_1, \operatorname{Arg}(z_0^2)) = \operatorname{Arg}(z_0^2)$. Since $\operatorname{Arg} z_1 \geqslant \operatorname{Arg} z_0$, $\operatorname{Arg}(z_1^2) \geqslant \operatorname{Arg}(z_0^2)$, so that $\operatorname{Arg} z_2 < \operatorname{Arg}(z_1^2)$, and therefore

$$\operatorname{Arg} z_1 < \operatorname{Arg}(z_2) < \operatorname{Arg}(z_1^2).$$

If we know only $\operatorname{Arg} z_0 \leqslant \operatorname{Arg} z_1 \leqslant 2 \operatorname{Arg} z_0$ we can conclude similarly that

$$\operatorname{Arg} z_1 \leqslant \operatorname{Arg}(z_2) \leqslant \operatorname{Arg}(z_1^2).$$

A formal induction argument using the same ideas can now be made to finish the proof of the lemma. ▮

Recall that Q_1 is the open first quadrant of $\mathbb{C}$, so that $\overline{Q_1} = \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0 \text{ and } \operatorname{Re} z \geqslant 0\}$. Hypotheses about arguments of numbers satisfying the QF recurrence allow conclusions about moduli in $\overline{Q_1} \setminus \{0\}$.

LEMMA 8.3. *Suppose that* $z_0, z_1 \in \overline{Q_1} \setminus \{0\}$ *and that* $\operatorname{Arg} z_0 \leqslant \operatorname{Arg} z_1 \leqslant \operatorname{Arg}(z_0^2)$. *For* $j > 1$ *define* $z_j = z_{j-1} + z_{j-2}^2$ *and assume that* $z_j \in \overline{Q_1}$ *for* $0 \leqslant j \leqslant N$ *with* $N \geqslant 1$. *Then* $|z_{j+1}| \geqslant |z_j|$ *for* $1 \leqslant j \leqslant N+1$.

Proof. The previous lemma shows that $z_j \neq 0$ for $0 \leqslant j \leqslant N+1$ and, if $\theta_j = \operatorname{Arg} z_j$ then $\theta_{j-1} \leqslant \theta_j \leqslant 2\theta_{j-1}$ for $1 \leqslant j \leqslant N+1$. Also, $\theta_{N+1} \leqslant \max(\theta_N, 2\theta_{N-1}) \leqslant \pi$. Lemma 8.1 implies strict inequality if $\theta_N \neq 2\theta_{N-1}$. If $\theta_N = 2\theta_{N-1}$, then since $z_N \in \overline{Q_1}$, $\theta_N \leqslant \frac{\pi}{2}$ so that $\theta_{N+1} < \pi$ in this case also.

Consider Fig. 6 for $1 \leqslant j \leqslant N+1$. The law of cosines gives

$$\begin{aligned} |z_{j+1}|^2 &= (\text{distance}(0, P_{j+1}))^2 \\ &= |z_j|^2 + (|z_{j-1}|^2)^2 - 2\,|z_j|\,|z_{j-1}|^2 \cos(2\pi - (\pi - \theta_j + 2\theta_{j-1})) \\ &= |z_j|^2 + |z_{j-1}|^4 + 2\,|z_j|\,|z_{j-1}|^2 \cos(2\theta_{j-1} - \theta_j). \end{aligned}$$

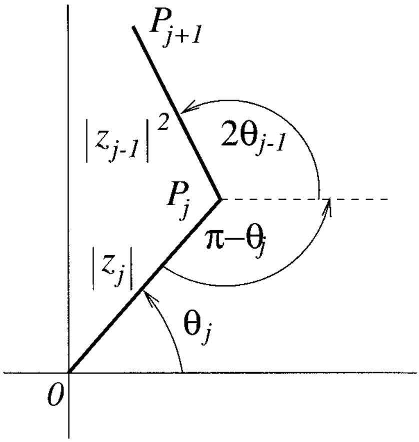

FIG. 6. The law of cosines.

But Lemma 8.2 shows that $2\theta_{j-1}-\theta_j \geqslant 0$ and that $2\theta_{j-1}-\theta_j \leqslant \theta_{j-1} \leqslant \theta_N \leqslant \frac{\pi}{2}$, so $\cos(2\theta_{j-1}-\theta_j) \geqslant 0$ for $1 \leqslant j \leqslant N+1$ and

$$|z_{j+1}|^2 \geqslant |z_j|^2 + |z_{j-1}|^4 + 2\,|z_j|\,|z_{j-1}|^2 \cos(\theta_{j-1}) \geqslant |z_j|^2 + |z_{j-1}|^4 \geqslant |z_j|^2$$

which completes the proof. ∎

If we know that $\theta_N < \frac{\pi}{2}$, so $\cos(\theta_N) = c > 0$, then

$$\begin{aligned} |z_{j+1}|^2 &\geqslant |z_j|^2 + |z_{j-1}|^4 + 2\,|z_j|\,|z_{j-1}|^2\,c \\ &= (|z_j| + c\,|z_{j-1}|^2)^2 + (1-c^2)\,|z_{j-1}|^4 \geqslant (|z_j| + c\,|z_{j-1}|^2)^2 \end{aligned}$$

and we have verified the following corollary:

COROLLARY 8.1. *If the hypotheses of the previous lemma hold and if also* $\theta_N < \frac{\pi}{2}$, *then* $|z_{j+1}| \geqslant |z_j| + c\,|z_{j-1}|^2$ *for* $1 \leqslant j \leqslant N+1$ *where* $c = \cos(\theta_N)$.

If $z_0, z_1 \in \mathbb{C}$ and $z_j = z_{j-1} + z_{j-2}^2$ for $j \geqslant 2$, then $z_j = q_j(z_0, z_1)$, a polynomial in z_0 and z_1 for $j \geqslant 2$. If $q_0(z_0, z_1) = z_0$ and $q_1(z_0, z_1) = z_1$, then $q_2(z_0, z_1) = z_1 + z_0^2$, $q_3(z_0, z_1) = z_1 + z_0^2 + z_1^2$, and generally $q_j(z_0, z_1) = q_{j-1}(z_0, z_1) + (q_{j-2}(z_0, z_1))^2$ for $j \geqslant 2$. If we choose $z_0 = z_1 = z$ and define $p_j(z) = q_j(z, z)$, then $p_j(z)$ is a polynomial in z whose coefficients are non-negative integers and $\deg(p_{2m}) = \deg(p_{2m+1}) = 2^m$ for integral $m \geqslant 0$ (here $\deg(P)$ denotes the degree of a polynomial P). $p_j(z)$ has no constant term and the coefficient of z is 1, so $p_j(z) = z + (\text{terms of degree greater than } 1)$.

DEFINITION 8.1. Suppose $N \geqslant 2$ is an integer and θ_* is a real number with $0 < \theta_* < \pi$. Then

$$U(N, \theta_*) = \{z \in \mathscr{H}_+ : \text{If } z_0 = z_1 = z \text{ and } z_{j+1} = z_j + z_{j-1}^2 \text{ for } j \geqslant 1, \\ \text{then } z_j \in \mathscr{H}_+ \text{ for } 1 \leqslant j \leqslant N \text{ and } \operatorname{Arg}(z_N) < \theta_*\}.$$

We could equivalently write

$$U(N, \theta_*) = \{z \in \mathscr{H}_+ : p_j(z) \in \mathscr{H}_+ \text{ for } 0 \leqslant j \leqslant N \text{ and } \operatorname{Arg}(p_N(z)) < \theta_*\}.$$

Lemma 8.2 shows that if $z \in \mathscr{H}_+$, $z = z_0 = z_1$, and $z_j \in \mathscr{H}_+ \cup \mathbb{R}_{\geqslant 0}$ for $1 \leqslant j \leqslant N$, then $z_j \neq 0$ for $1 \leqslant j \leqslant N$ and $\operatorname{Arg} z_{j-1} < \operatorname{Arg} z_j < 2 \operatorname{Arg} z_{j-1}$ for $2 \leqslant j \leqslant N$. So certainly each $z_j \in \mathscr{H}_+$ for $0 \leqslant j \leqslant N-1$. If $\operatorname{Arg} z_N < \theta_*$ also, then $z \in U(N, \theta_*)$. Thus we can also write

$$U(N, \theta_*) = \{z \in \mathscr{H}_+ : p_j(z) \in \mathscr{H}_+ \cup \mathbb{R}_{\geqslant 0} \text{ for } 1 \leqslant j \leqslant N \\ \text{and } \operatorname{Arg}(p_N(z)) < \theta_*\}.$$

The following lemma will play a significant part in creating a holomorphic analog of the function L. It is similar in purpose to the simpler lemmas in Section 3.

LEMMA 8.4. *Suppose that $0 < \theta_* < \pi$ and that $N \geqslant 2$ is an integer. If $w \in \mathscr{H}_+$ and $\operatorname{Arg} w < \theta_*$, there is a unique complex number $z \in U(N, \theta_*)$ such that $p_N(z) = w$. Furthermore, $p'_N(z) \neq 0$.*

Remark. Before proving this result, note that if $w \in \mathbb{R}_{>0}$ and if $p_N(z) = w$ for some $z \in \mathscr{H}_+ \cup \mathbb{R}_{\geqslant 0}$ so that $p_j(z) \in \mathscr{H}_+ \cup \mathbb{R}_{\geqslant 0}$ when $1 \leqslant j \leqslant N$, then by Lemma 8.2, $z \in \mathbb{R}_{>0}$: otherwise $\operatorname{Arg} z > 0$ so $\operatorname{Arg} p_N(z) > 0$. Since the coefficients of $p_N(z)$ are all positive integers, and $p_N(0) = 0$, we see that $p_N(z) = w$ will have exactly one solution and that $p'_N(z) > 0$ for all $z \in \mathbb{R}_{>0}$.

Proof. We first check that $U(N, \theta_*)$ is nonempty. For sufficiently small z, $p_j(z)$ is close to z. This allows us to conclude that $U(N, \theta_*)$ is nonempty. Less concisely, given $\varepsilon > 0$, there is $\rho(\varepsilon) > 0$ so that if $|z| < \rho(\varepsilon)$ then $|p_j(z) - z| < \varepsilon\,|z|$ for $1 \leqslant j \leqslant N$. Now fix θ with $0 < \theta < \theta_*$ and take z with $\operatorname{Arg} z = \theta$. Select ε small enough so that the disc $\{w \in \mathbb{C} : |w - z| < \varepsilon\,|z|\}$ lies inside the wedge $\{w \in \mathscr{H}_+ : \operatorname{Arg} w < \theta_*\}$. If additionally $|z| < \rho(\varepsilon)$, certainly $z \in U(N, \theta_*)$. Since each of the p_j's is continuous, $U(N, \theta_*)$ is open.

We will also need information about $\partial U(N, \theta_*)$. Each p_j has positive integer coefficients, so $\partial U(N, \theta_*) \supset \mathbb{R}_{\geqslant 0}$: z's in $\mathscr{H}_+$ close to the nonnegative real axis must be in $U(N, \theta_*)$ because $\operatorname{Arg}(p_j(re^{i\theta})) \to 0^+$ for r and j fixed as $\theta \to 0^+$. Now suppose that $z \notin \mathbb{R}_{\geqslant 0}$ and $z \in \partial U(N, \theta_*)$. z must be the limit of a sequence $\{z_k\}$ in $U(N, \theta_*)$. We know that $z_k \in \mathscr{H}_+$, and, for $1 \leqslant j \leqslant N$, $p_j(z_k) \in \mathscr{H}_+$ and $0 < \operatorname{Arg}(p_j(z_k)) < \theta_*$. Since $z \notin \mathbb{R}_{\geqslant 0}$, $\operatorname{Arg} z_k \to \operatorname{Arg} z > 0$ so that $0 < \operatorname{Arg} z \leqslant \theta_*$. Lemma 8.2 implies that $p_2(z) \neq 0$ and $0 < \operatorname{Arg} p_2(z) < 2\pi$, so $\operatorname{Arg}(p_2(z_k)) \to \operatorname{Arg}(p_2(z))$. Thus $0 < \operatorname{Arg}(z) \leqslant \operatorname{Arg}(p_2(z)) \leqslant \theta_*$. We may apply Lemmas 8.1 and 8.2 inductively together with continuity of argument (applied to sequences of non-zero complex numbers with non-zero limits all contained in the interior of our argument's domain!) to conclude that if $z \in \partial U(N, \theta_*)$ and $z \notin \mathbb{R}_{\geqslant 0}$ then $0 < \operatorname{Arg} z \leqslant \operatorname{Arg}(p_2(z)) \leqslant \cdots \leqslant \operatorname{Arg}(p_N(z)) \leqslant \theta_*$. If $\operatorname{Arg}(p_N(z)) < \theta_*$, then $z \in U$. Therefore, if $z \in \partial U(N, \theta_*)$, either $z \in \mathbb{R}_{\geqslant 0}$ or $p_j(z) \neq 0$ for $1 \leqslant j \leqslant N$ and $0 < \operatorname{Arg} z \leqslant (p_2(z)) \leqslant \cdots \leqslant \operatorname{Arg}(p_N(z)) = \theta_*$.

Now suppose $w \in \mathscr{H}_+$ with $\operatorname{Arg} w < \theta_*$. Select $R > 0$ so that $|p_N(z)| > |w|$ when $|z| > R$. Let $U(N, \theta_*)_R = \{z \in U(N, \theta_*) : |z| < R\}$. $U(N, \theta_*)_R$ is a bounded open set. We claim that

$$\deg(p_N, U(N, \theta_*)_R, tw) = 1 \tag{$*$}$$

for $0 < t \leqslant 1$.

We first must show that $p_N(z) \neq tw$ for $z \in \partial U(N, \theta_*)_R$ and $0 < t \leqslant 1$. $\deg(p_N, U(N, \theta_*)_R, tw)$ will then be defined and will not depend on t by the homotopy property of degree theory. If $z \in \partial U(N, \theta_*)_R$ then either (i) $|z| = R$ or (ii) $z \in \mathbb{R}_{\geqslant 0}$ or (iii) $z \in \mathscr{H}_+$ and $\operatorname{Arg} p_N(z) = \theta_*$.

(i) If $|z| = R$ then $|p_N(z)| > |w| \geqslant |tw|$ so $p_n(z) \neq tw$.

(ii) If $z \in \mathbb{R}_{\geqslant 0}$ then $p_N(z) \in \mathbb{R}_{\geqslant 0}$ so $p_N(z) \neq tw$.

(iii) If $\operatorname{Arg}(p_N(z)) = \theta_*$, then $\operatorname{Arg}(p_N(z)) > \operatorname{Arg}(w) = \operatorname{Arg}(tw)$, so $p_n(z) \neq tw$.

Suppose we prove that for sufficiently small $t > 0$ the equation $p_N(z) = tw$ has exactly one solution $z = z(t) \in U(N, \theta_*)$ and that $p'_N(z(t)) \neq 0$. Then we see that $\deg(p_N, U(N, \theta_*)_R, tw) = \deg(p_N, U(N, \theta_*)_R, w) = 1$ and properties of degree theory for holomorphic maps imply that $p_N(z) = w$ has a unique solution $z_R \in U(N, \theta_*)_R$ (and therefore a unique solution in $U(N, \theta_*)$) and that $p'_N(z) \neq 0$.

So we study the equation $p_n(z) = tw$ for small t. We apply the implicit function theorem to the function $F(z, t) = p_N(z) - tw$ for $z \in \mathbb{C}$ and $t \in \mathbb{R}$. Certainly $F(0, 0) = 0$ and $(\partial F/\partial z)(0, 0) = 1$. The implicit function theorem implies there are $\rho > 0$ and $\varepsilon > 0$ so that if $|t| < \varepsilon$ the map $z \mapsto F(z, t)$ is one-one on $B_\rho(0) = \{z\colon |z| < \rho\}$ and for each t with $|t| < \varepsilon$, the equation $F(z, t) = 0$ has a unique solution $z(t) \in B_\rho(0)$. The map $t \mapsto z(t)$ is C^∞ and $z'(0) = w$. Therefore $z(t) = tw + O(t^2)$ as $t \to 0$. This is enough to conclude that $z(t) \in U(N, \theta_*)$ for sufficiently small t, because $p_j(z(t)) = tw + O(t^2)$ for $1 \leqslant j \leqslant N$ (again using the specific form of the polynomials p_j). Also

$p'_N(z(t)) = 1 + O(t)$, so $p'_N(z(t)) \neq 0$ for $t > 0$ and t small. Therefore we can choose positive $\varepsilon_1 < \varepsilon$ so that $z(t) \in U(N, \theta_*)$ and $p'_N(z(t)) \neq 0$ for $0 < t < \varepsilon_1$.

We can thus conclude that $\deg(p_N, U(N, \theta_*), tw) \geqslant 1$ for sufficiently small positive t. We complete the proof of the lemma by showing that there is $\varepsilon_2 > 0$ so that if $0 < t < \varepsilon_2 \leqslant \varepsilon_1$ and $F(z, t) = 0$ for some $z \in U(N, \theta_*)$, then $|z| < \rho$. This forces z to equal the $z(t)$ previously exhibited and the degree must equal 1. If $0 < \operatorname{Arg} w \leqslant \frac{\pi}{2}$ then Lemma 8.3 applies and we can conclude that $|z| \leqslant |p_2(z)| \leqslant \cdots \leqslant |p_N(z)| = |tw|$. Take positive $\varepsilon_2 < \varepsilon_1$ so that if $|t| < \varepsilon_2$, then $|tw| < \rho$.

The analysis of the case when $\frac{\pi}{2} < \operatorname{Arg} w < \theta_*$ is more extended. We suppose that $p_N(z) = tw$ where $0 < t < \varepsilon_2$ and $\varepsilon_2 < \varepsilon_1$ will be selected later. Let $z_j = p_j(z)$ for $1 \leqslant j \leqslant N$. If $\theta_j = \operatorname{Arg} z_j$ then $\theta_{j-1} < \theta_j < 2\theta_{j-1}$ for $2 \leqslant j \leqslant N$, and $\theta_N = \operatorname{Arg} w$. We seek to bound $|z_{j-1}|$ by an appropriate multiple of $|z_j|$. As in the discussion of Lemma 8.3, we write:

$$|z_j|^2 = |z_{j-1}|^2 + |z_{j-2}|^4 + 2\,|z_{j-1}|\,|z_{j-2}|^2 \cos(2\theta_{j-2} - \theta_{j-1}). \qquad (**)$$

Certainly $0 \leqslant 2\theta_{j-2} - \theta_{j-1} \leqslant \theta_{j-2} \leqslant \theta_{N-2} \leqslant \operatorname{Arg} w < \pi$ when $2 \leqslant j \leqslant N$. If $2\theta_{j-2} - \theta_{j-1} \leqslant \frac{\pi}{2}$, then $(**)$ implies $|z_j| \geqslant \sqrt{|z_{j-1}|^2 + |z_{j-2}|^4} \geqslant |z_{j-1}|$. This will be true if $\operatorname{Arg} w \leqslant \frac{\pi}{2}$ or if just $\theta_{N-2} \leqslant \frac{\pi}{2}$. If $\frac{\pi}{2} < 2\theta_{j-2} - \theta_{j-1}$ (so $\frac{\pi}{2} < \theta_{j-2}$) we complete the square in $(**)$:

$$\begin{aligned} |z_j|^2 &= (|z_{j-1}| + |z_{j-2}|^2 \cos(2\theta_{j-2} - \theta_{j-1}))^2 \\ &\quad + (1 - (\cos(2\theta_{j-2} - \theta_{j-1}))^2)\,|z_{j-2}|^4 \\ &\leqslant (1 - (\cos(\operatorname{Arg}(w)))^2)\,|z_{j-2}|^4 = (1 - \kappa^2)\,|z_{j-2}|^4. \end{aligned} \qquad (***)$$

Here $\kappa = \cos(\operatorname{Arg}(w))$. Because $\frac{\pi}{2} < \theta_{j-2}$ and $\theta_{j-2} \leqslant \operatorname{Arg}(w) < \pi$, we know $0 > \kappa > -1$ and $1 > 1 - \kappa^2 > 0$. Therefore $|z_j| \geqslant \sqrt{1-\kappa^2}\,|z_{j-1}|^2$. But $(***)$ also implies

$$\begin{aligned} |z_j| &\geqslant |(|z_{j-1}| + |z_{j-2}|^2 \cos(2\theta_{j-2} - \theta_{j-1}))| \\ &\geqslant |z_{j-1}| - |\kappa|\,|z_{j-2}|^2 \geqslant |z_{j-1}| - \frac{|\kappa|}{\sqrt{1-\kappa^2}}\,|z_j| \end{aligned}$$

so that when $2\theta_{j-2} - \theta_{j-1} > \frac{\pi}{2}$ we obtain

$$\overbrace{\left(\frac{\sqrt{1-\kappa^2} + |\kappa|}{\sqrt{1-\kappa^2}}\right)}^{\kappa_1} |z_j| \geqslant |z_{j-1}|.$$

If $\kappa_2 = \max(\kappa_1, 1)$ (the first entry takes care of the case $\operatorname{Arg} z \in (\frac{\pi}{2}, \pi)$ and the second, $\operatorname{Arg} z \in (0, \frac{\pi}{2}]$)) and $2 \leqslant j \leqslant N$, then

$$\kappa_2 \, |z_j| \geqslant |z_{j-1}|$$

which results in

$$(\kappa_2)^N \, |z_N| \geqslant |z_1| = |z_0|.$$

Therefore

$$|z| = |z_0| \leqslant (\kappa_2)^N \, |z_N| = (\kappa_2)^N \, |tw| \leqslant (\kappa_2)^N \, \varepsilon_2 \, |w|$$

so $|z| < \rho$ for ε_2 sufficiently small. ∎

Let $\mathbb{C}_- = \mathbb{C} \backslash \mathbb{R}_{\leqslant 0}$. $\mathbb{C}_-$, a "slit" plane, is an open subset of $\mathbb{C}$ and invariant under conjugation. Other slits in the complex numbers could possibly be used to prove results similar to those here. This choice seems simplest since the polynomials p_n have real coefficients and their symmetry acting on $\mathbb{C}_-$ helps to prove the major result of this section: that $\mathbb{C}_-$ is the domain of a holomorphic extension of the function L defined earlier. We need preliminary work before this result is verified.

Let $\mathscr{U}_N = \{z \in \mathscr{H}_+ : p_j(z) \in \mathscr{H}_+ \text{ for } 0 \leqslant j \leqslant N\}$. $\mathscr{U}_N$ is the union of the sets $U(N, \theta_*)$ for $0 < \theta_* < \pi$. We let $G_N = \mathscr{U}_N \cup \overline{\mathscr{U}_N} \cup \mathbb{R}_{>0}$, where $\overline{\mathscr{U}_N} = \{\bar{z} \colon z \in \mathscr{U}_N\}$.

Lemma 8.5. *G_N is an open subset of $\mathbb{C}_-$. For each $w \in \mathbb{C}_-$, there is a unique $z \in G_N$ so that $p_N(z) = w$, and $p'_N(z) \neq 0$.*

Proof. Since $\mathscr{U}_N$ and its conjugate are open in $\mathbb{C}_-$, we need only check that $\mathbb{R}_{>0}$ is a subset of the interior of G_N. But given $q(z)$, a non-zero polynomial whose non-zero coefficients are real and positive, there is ψ with $0 < \psi < \pi$ so that if $z \in \mathscr{H}_+$ and $\arg z < \psi$ then $q(z) \in \mathscr{H}_+$. This can be seen, successively, for monomials, positive multiples of monomials, and sums of such terms. If we then define the wedge K_ψ by $K_\psi = \{z \in \mathbb{C} : |\arg z| < \psi\}$, we can surely find ψ so that $\mathbb{R}_{>0} \subset K_\psi \subset G_N$.

If $w \in \mathscr{H}_+$, Lemma 8.4 shows that there is exactly one $z \in \mathscr{U}_N$ with $p_N(z) = w$ and $p'_N(z) \neq 0$. Conjugation shows that if $w \in \mathscr{H}_-$, there is exactly one $z \in \overline{\mathscr{U}_N}$ with $p_N(z) = w$ and $p'_N(z) \neq 0$. We finish the proof by noting that the desired behavior for p_N on $\mathbb{R}_{>0}$ was exactly described in the remarks following the statement of Lemma 8.4. ∎

The restriction of p_N to the domain G_N will be called $\tilde{p}_N$. We define $q_N \colon \mathbb{C}_- \to G_N$ to be the inverse of $\tilde{p}_N$. q_N is a biholomorphic mapping of $\mathbb{C}_-$ to G_N. If $j \geqslant -N$, $h_{j,N}\colon \mathbb{C}_- \to \mathbb{C}$ will be defined by $h_{j,N}(z) = p_{N+j}(q_N(z))$. We also know that

$$(h_{j,N}(z))^2 + h_{j+1,N}(z) = h_{j+2,N}(z)$$

for all $j \geqslant -N$ and $h_{j,N}(z) \in \mathbb{C}_-$ for $-N \leqslant j \leqslant 0$. Of course $h_{j,N}(\mathbb{R}_{>0}) \subset \mathbb{R}_{>0}$ for $j \geqslant -N$ and $h_{j,N}(\mathscr{H}_+) \subset \mathscr{H}_+$, and $h_{j,N}(\mathscr{H}_-) \subset \mathscr{H}_-$ for $j \geqslant -N$. When $-N \leqslant j < 0$ we also know:

if $z \in \mathscr{H}_+$, $0 < \arg(h_{j,N}(z)) \leqslant \arg(h_{j+1,N}(z)) \leqslant 2\arg(h_{j,N}(z))$;

if $z \in \mathscr{H}_-$, $2\arg(h_{j,N}(z)) \leqslant \arg(h_{j+1,N}(z)) \leqslant \arg(h_{j,N}(z)) < 0$.

Suppose θ_* is between 0 and π, and $\overline{K_{\theta_*}} = \{z \in \mathbb{C} : |\arg z| \leqslant \theta_*\}$. The proof of Lemma 8.4 provides $\kappa_2 > 0$ so that

$$|h_{j,N}(z)| \leqslant \kappa_2^{|j|}\,|z|$$

when $-N \leqslant j \leqslant 0$ and $z \in \overline{K_{\theta_*}} \backslash \{0\}$ and

$$|h_{j-1,N}(z)| \leqslant \kappa_2\,|h_{j,N}(z)|$$

when $-N < j \leqslant 0$ and $z \in \overline{K_{\theta_*}} \backslash \{0\}$.

Any compact subset C of $\mathbb{C}_-$ must be contained in some $\overline{K_{\theta_*}}$. Suppose that C is such a set and $j \in \mathbb{Z}$ is fixed. The preceding inequalities and the QF recurrence allow us to conclude that there is a constant $M(j, C)$ depending on j and C so that for all $N \geqslant 2$ with $-N \leqslant j$ and all $z \in C$,

$$|h_{j,N}(z)| \leqslant M(j, C).$$

We may then use Montel's Theorem and Cantor diagonalization (since $\mathbb{C}_-$ is $\sigma-$compact) to conclude that there is an increasing subsequence $\{N_k\}$ such that for each $j \in \mathbb{Z}$ and $z \in \mathbb{C}_-$

$$\lim_{k \to \infty} h_{j,N_k}(z) = h_j(z)$$

exists. The convergence is uniform in z for z in any compact subset of $\mathbb{C}_-$ and the functions h_j are therefore holomorphic in $\mathbb{C}_-$. The use of Montel's Theorem at this stage of the argument is somewhat analogous to the intersection of a sequence of nested closed intervals in the proof of Theorem 3.1.

Now we consider the properties of the limit functions, h_j, and their properties, most of which are simple inheritances from the $h_{j,N}$'s. If we define $h_j(0) = 0$ then the inequalities show that h_j is continuous in $\mathbb{C}_- \cup \{0\}$. We also see that

$$|h_{j-1}(z)| \leqslant \kappa_2\,|h_j(z)|$$

for all $j \leqslant 0$ and $z \in \overline{K_{\theta_*}}$. The limits certainly preserve the recurrence so

$$(h_j(z))^2 + h_{j+1}(z) = h_{j+2}(z)$$

holds for all $z \in \mathbb{C}_-$ and all $j \in \mathbb{Z}$.

Suppose $h_{j_*}(z) = 0$ for some $j_* \leqslant 0$ and $z \in \mathbb{C}_-$. Since $h_0(z) = z$, $j_* < 0$. But then the preceding inequality implies that $h_j(z) = 0$ for all $j \leqslant j_*$. The recurrence then shows that $h_j(z) = 0$ for all j, contradicting $h_0(z) = z \neq 0$.

If $z \in \mathscr{H}_+$ and $j \leqslant 0$, then $h_j(z) \in \overline{\mathscr{H}_+}$ by taking limits. But since $h_j(z) \neq 0$ for $j \leqslant 0$ we may also take limits in the inequalities for arguments to get

$$0 \leqslant \arg h_{j-1}(z) \leqslant \arg h_j(z) \leqslant 2 \arg h_{j-1}(z)$$

when $-N < j \leqslant 0$. Since $h_0(z) = z$ we see that $0 \leqslant \arg(h_j(z)) < \pi$. If $h_j(z) \in \mathbb{R}_{>0}$ then $h_{j-1}(z) \in \mathbb{R}_{>0}$ also, implying by the recurrence that $h_0(z) = z \in \mathbb{R}_{>0}$ which is incorrect. Therefore $\arg h_j(z) > 0$ so $h_j(z) \in \mathscr{H}_+$.

Symmetry under conjugation is preserved by limits, so $h_j(\bar{z}) = \overline{h_j(z)}$ for all $j \in \mathbb{Z}$ and $z \in \mathbb{C}_-$. Thus $h_j(\mathscr{H}_-) \subset \mathscr{H}_-$ for $j \leqslant 0$ and

$$2 \arg(h_{j-1}(z)) \leqslant \arg(h_j(z)) \leqslant \arg(h_{j-1})(z)) < 0$$

for all $j < 0$ and $z \in \mathscr{H}_-$.

We know that $h_{j,N}(\mathbb{R}_{>0}) \subset \mathbb{R}_{>0}$ for all $j \geqslant -N$ so $h_j(\mathbb{R}_{>0}) \subset \mathbb{R}_{\geqslant 0}$. But if $h_{j_*}(x) = 0$ for $x \in \mathbb{R}_{>0}$, the recurrence shows that $h_j(x) = 0$ for all j which we know to be false for negative j. Thus $h_j(\mathbb{R}_{>0}) \subset \mathbb{R}_{>0}$ for all $j \in \mathbb{Z}$.

If $z \in \mathbb{C}_-$, the sequence $\{h_j(z)\}_{j \in \mathbb{Z}}$ which we have begun to investigate has properties interesting enough to name distinctly.

Definition 8.2. A doubly infinite sequence $\mathbf{Z} = \{z_j\}_{j \in \mathbb{Z}}$ is an *argument increasing recurrence sequence* through $z \in \mathbb{C}_-$ if all z_j's are in $\mathbb{C}_-$ for $j < 0$, $z_0 = z$, $z_j = z_{j-1} + z_{j-2}^2$ for $j \in \mathbb{Z}$, and one of the following occurs:

(i) $z_j \in \mathscr{H}_+$ for all $j \leqslant 0$ and $\arg z_{j-1} \leqslant \arg z_j \leqslant 2 \arg z_{j-1}$ for all $j \leqslant 0$.

(ii) $z_j \in \mathbb{R}_{>0}$ for all $j \leqslant 0$.

(iii) $z_j \in \mathscr{H}_-$ for all $j \leqslant 0$ and $2 \arg z_{j-1} \leqslant \arg z_j \leqslant \arg z_{j-1}$ for all $j \leqslant 0$.

We will sometimes say that $\mathbf{Z}$ is an *AIR* sequence through z. Any simple extension of this definition to the negative reals will need to break the symmetry between the upper and lower half-planes, since we have seen (Lemma 3.1) that there are no sequences whose tails (the z_j's for $j < 0$) are all negative real numbers.

Proposition 8.1. *Suppose $\mathbf{Z} = \{z_j\}_{j \in \mathbb{Z}}$ is a sequence of complex numbers with $z_j \in \mathscr{H}_+$ for all j and that $0 \leqslant \arg z_{j-1} \leqslant \arg z_j$ for $j \leqslant 0$ and $z_j =$*

$z_{j-1}+z_{j-2}^2$ *for all* $j \in \mathbb{Z}$. *Then* $\lim_{j\to-\infty} \arg z_j = 0$ *and* $\sum_{j=-\infty}^{0} |z_j|^2 < \infty$, *so* $\lim_{j\to-\infty} z_j = 0$.

Remark. We shall frequently prove results for all *AIR* sequences by verifying them for the upper half-plane using proofs which can be reflected by complex conjugation. The result above verifies that the tail of any *AIR* sequence is in ℓ^2 and must therefore approach 0. Since the arguments decrease to 0 in the upper half-plane, Lemma 8.3 eventually applies, and therefore the tail of an *AIR* sequence eventually also decreases in modulus: that is, there exists j_* so that $|z_j| \leqslant |z_{j+1}|$ for all $j \leqslant j_*$.

Proof. Let $\rho_j = |z_j|$ and $\theta_j = \arg z_j$. Since $0 \leqslant \theta_{j-1} \leqslant \theta_j < \pi$ for $j \leqslant 0$, $\lim_{j\to-\infty} \theta_j = \mu \geqslant 0$ exists. The recurrence can be written as $z_{j-2}^2 = z_j - z_{j-1}$. With $n < m \leqslant 0$, sum and cancel so that

$$\sum_{j=n-2}^{m-2} z_j^2 = z_m - z_{n-1}. \tag{$\dagger$}$$

Multiply by $e^{-i\mu}$ to get

$$\sum_{j=n-2}^{m-2} \rho_j^2 e^{i(2\theta_j-\mu)} = \rho_m e^{i(\theta_m-\mu)} - \rho_{n-1} e^{i(\theta_{n-1}-\mu)}.$$

Now suppose $\mu > 0$. We show that this leads to a contradiction. Select $\delta > 0$ so that $\delta < \frac{1}{2}\mu$ and $\mu + 2\delta < \pi$. Select $m \leqslant 0$ so that $\mu \leqslant \theta_j < \mu + \delta$ for all $j \leqslant m$. We know that $\mu \leqslant 2\theta_j - \mu < \mu + 2\delta$, so that $\sin(2\theta_j - \mu) \geqslant \min(\sin\mu, \sin(\mu+2\delta) = t > 0$. We then estimate the imaginary parts of the previous equation:

$$t \sum_{j=n-2}^{m-2} \rho_j^2 \leqslant \sum_{j=n-2}^{m-2} \rho_j^2 \sin(2\theta_j - \mu) \leqslant \rho_m \sin(\theta_m - \mu)$$

since $\sin(\theta_{n-1} - \mu)$ is always positive. Let $n \to -\infty$ and conclude that $\sum_{j=-\infty}^{m} \rho_j^2 < \infty$. Thus $\lim_{j\to-\infty} \rho_j = 0$.

Multiply ($\dagger$) by $e^{-i\theta_m}$ and let $n \to -\infty$. Then

$$\sum_{j=\infty}^{m-2} \rho_j^2 e^{i(2\theta_j-\theta_m)} = \rho_m.$$

Here $2\theta_j - \theta_m$ must be between $\mu - \delta$ and $\mu + 2\delta$ which are both between 0 and π. If $t_1 = \min(\sin(\mu-\delta, \mu+2\delta)$ then $t_1 > 0$, and, again taking imaginary parts,

$$t_1 \sum_{j=\infty}^{m-2} \rho_j^2 \leqslant \sum_{j=\infty}^{m-2} \rho_j^2 \sin(2\theta_j - \theta_m) = \operatorname{Im} \rho_m = 0.$$

This can only occur if all of the ρ_j's are 0 for $j \leqslant m-2$. The recurrence equation then implies that all z_j's are 0, so μ must be 0.

We still must show that the tail of $\mathbf{Z}$ is in ℓ^2. Since $\mu = 0$ we may now find m so that $0 \leqslant 2\theta_{m-2} < \frac{\pi}{2}$ and $0 \leqslant \theta_m < \frac{\pi}{2}$. The real part of (†) then yields

$$0 \leqslant \cos(2\theta_{m-2}) \sum_{j=n-2}^{m-2} \rho_j^2 \leqslant \sum_{j=n-2}^{m-2} \rho_j^2 \cos(2\theta_j)$$

$$= \rho_m \cos\theta_m - \rho_{n-1} \cos\theta_{n-1} \leqslant \rho_m \cos\theta_m$$

which gives, as $n \to \infty$, the estimate $\sum_{j=n-2}^{m-2} \rho_j^2 < (\rho_m \cos\theta_m / \cos(2\theta_{m-2}))$. ∎

We have proven that $\{h_j(z)\}_{j \in \mathbb{Z}}$ is an *AIR* sequence through any $z \in \mathbb{C}_-$. Theorem 3.2 shows that there is a *unique AIR* sequence through any $x \in \mathbb{R}_{>0}$. We use this uniqueness and analytic continuation on the connected open set $\mathbb{C}_-$ to derive useful relationships among the h_j's.

If $x \in \mathbb{R}_{>0}$, $\{x_j\}$ will denote the unique *AIR* sequence through x, so $x_0 = x$. Therefore $x_j = h_j(x)$. But $\{x_{j-1}\}$ is the unique *AIR* sequence through $x_{-1} \in \mathbb{R}_{>0}$, so $h_j(x) = h_{j+1}(x_{-1}) = h_{j+1}(h_{-1}(x))$. We can repeat this to conclude that $h_j(x) = h_{-1}^{[|j|]}(x)$ for all $x \in \mathbb{R}_{>0}$ and $j < 0$ (here $h_{-1}^{[|j|]}$ represents the composition of h_{-1} with itself $|j|$ times). h_{-1} and all h_j for $j < 0$ map $\mathbb{C}_-$ into itself holomorphically. Therefore we may conclude that $h_j(z) = h_{-1}^{[|j|]}(z)$ for all $z \in \mathbb{C}_-$ and all integers $j < 0$.

Similar reasoning allows the conclusions $h_1(h_{-1}(x)) = x$ and $h_{-1}(h_1(x)) = x$ for $x \in \mathbb{R}_{>0}$. The domain of the holomorphic function $h_1 \circ h_{-1}$ is $\mathbb{C}_-$, a connected open set, so that $h_1(h_{-1}(z)) = z$ for all $z \in \mathbb{C}_-$ and h_{-1} must be one-to-one. However, $h_1(\mathbb{C}_-)$ may not be contained in $\mathbb{C}_-$. We may therefore conclude only that $h_{-1}(h_1(z)) = z$ for all $z \in h_{-1}(\mathbb{C}_-)$, a connected open subset of $\mathbb{C}_-$ containing $\mathbb{R}_{>0}$.

The following existence result for *AIR* sequences is now proved.

THEOREM 8.1. *There exists an injective holomorphic map $h_{-1}\colon \mathbb{C}_- \to \mathbb{C}_-$ so that h_{-1} maps $\mathscr{H}_+$ to $\mathscr{H}_+$ and $\mathbb{R}_{>0}$ to $\mathbb{R}_{>0}$, and $h_{-1}(\bar{z}) = \overline{h_{-1}(z)}$. If $h_j(z) = h_{-1}^{[|j|]}(z)$ for $j < 0$, $h_0(z) = z$, and $h_j(z)$ for $j > 0$ is defined by $h_j(z) = h_{j-1}(z) + (h_{j-2}(z))^2$, then h_j is holomorphic on $\mathbb{C}_-$ for all $j \in \mathbb{Z}$ and $h_j(\mathbb{C}_-) \subset \mathbb{C}_-$ for $j < 0$. Also, $h_1(h_{-1}(z)) = z$ for all $z \in \mathbb{C}_-$. For each $z \in \mathbb{C}_-$, $\{h_j(z)\}_{j \in \mathbb{Z}}$ is an AIR sequence through z.*

Techniques involving continued fractions can also be used to study the existence of *AIR* sequences, and provide a description of h_{-1} as a locally uniform limit of a sequence of linear fractional transformations. Since *AIR* sequences through elements of $\mathbb{R}_{>0}$ are unique, the function L defined in

Section 3. must coincide with h_1, which maps $\mathbb{R}_{>0}$ to itself. Restrictions of holomorphic functions to $\mathbb{R}_{>0}$ are real analytic, so the corollary below is verified. Direct proof of this using the formulas in Sections 3 and 4. seems difficult.

COROLLARY 8.2. *The function L is real analytic on* $\mathbb{R}_{>0}$.

We do not know if there is exactly one *AIR* sequence through each $z \in \mathbb{C}_-$ but results on continued fractions imply uniqueness for some non-real z's. We present the uniqueness theorem below, but first quote the needed information about continued fractions.

THEOREM 8.2. *Suppose* $D = \{z \in \mathbb{C} : |z| - \mathrm{Re}(z) \leqslant \frac{1}{2}\}$ *and suppose that C is any compact subset of D. Let* $\{\gamma_j\}_{j \geqslant 2}$ *be any sequence of complex numbers in C. Then for* $k \geqslant 2$, $[[\gamma_2, \gamma_3, ..., \gamma_k]]$ *is non-zero, and* $[[\gamma_2, \gamma_3, ..., \gamma_k]]^{-1} \in V = \{z \in \mathbb{C} : |z-1| \leqslant 1, z \neq 0\}$ *and* $[[\gamma_2, \gamma_3, ..., \gamma_k]]^{-1}$ *converges as* $k \to \infty$ *to an element of V. This convergence is uniform for all sequences* $\{\gamma_j\}_{j \geqslant 2}$ *with elements in C.*

The theorem is a combination of various statements about continued fractions in [21]: Theorem 14.2 (p. 58 of [21]), Theorem 14.3 (p. 60), and Theorem 18.1 (p. 78). It contains a precise statement of Worpitzky's Theorem used earlier. We now state and prove our result about uniqueness of *AIR* sequences.

THEOREM 8.3. *Suppose that* $z \in \mathbb{C}$ *satisfies* $\mathrm{Re}\, z \geqslant 0$, $z \neq 0$ *and* $|z| - \mathrm{Re}(z) \leqslant \frac{1}{4}$. *Then there is a unique AIR sequence through z.*

Proof. We consider here the case $\mathrm{Im}\, z \geqslant 0$. Theorem 8.1 shows that there is at least one *AIR* sequence through z. Suppose there are two distinct *AIR* sequences through z, $\{z_j\}_{j \in \mathbb{Z}}$ and $\{\hat{z}_j\}_{j \in \mathbb{Z}}$. We know $z_0 = \hat{z}_0 = z$.

We establish some notation. For $j \leqslant 0$, let $z_j = \rho_j e^{i\theta_j}$, $\theta_j = \arg z_j$, $\rho_j = |z_j|$, $\hat{z}_j = \hat{\rho}_j e^{i\hat{\theta}_j}$, $\hat{\theta}_j = \arg \hat{z}_j$, and $\hat{\rho}_j = |\hat{z}_j|$. Since we know $z \in \overline{Q_1} \backslash \{0\}$, all of the indicated arguments exist, and $\rho_{j-1} \leqslant \rho_j$, $0 \leqslant \theta_{j-1} \leqslant \theta_j \leqslant 2\theta_{j-1}$ for $j \leqslant 0$ (with analogous statements also valid for $\hat{\rho}_j$ and $\hat{\theta}_j$). We have further assumed that

$$|z| - \mathrm{Re}\, z = \rho_0 - \rho_0 \cos \theta_0 = \hat{\rho}_0 - \hat{\rho}_0 \cos \hat{\theta}_0 \leqslant \tfrac{1}{4}$$

so that for $j \leqslant 0$ we know

$$|z_j| - \mathrm{Re}\, z_j = \rho_j (1 - \cos \theta_j) \leqslant \rho_0 (1 - \cos \theta_0) \leqslant \tfrac{1}{4}$$

and similarly

$$|\hat{z}_j| - \mathrm{Re}\, \hat{z}_j \leqslant \tfrac{1}{4}.$$

Add these inequalities and estimate $|z_j+\hat{z}_j|$ by $|z_j|+|\hat{z}_j|$ to see that

$$|z_j+\hat{z}_j|-\mathrm{Re}(z_j+\hat{z}_j)\leqslant \tfrac{1}{2}$$

for $j\leqslant 0$. If $\gamma_j=z_{-j}+\hat{z}_{-j}$ for $j\geqslant 0$, each entry of the sequence $\{\gamma_j\}_{j\in\mathbb{N}}$ is in the set D of Theorem 8.2 and $\lim_{j\to\infty}\gamma_j=0$ by Proposition 8.1. Therefore $\{\gamma_j\}_{j\in\mathbb{N}}$ lies in some compact subset C of D.

The recurrence relation satisfied by both sequences shows that

$$\hat{z}_j-z_j=\hat{z}_{j-1}-z_{j-1}+(\hat{z}_{j-2}-z_{j-2})(\hat{z}_{j-2}+z_{j-2})$$

and we use this to compare the sequences.

If $j\leqslant 0$ and $\hat{z}_{j-1}\neq z_{j-1}$, then

$$\frac{\hat{z}_j-z_j}{\hat{z}_{j-1}-z_{j-1}}=1+\left(\frac{\hat{z}_{j-2}-z_{j-2}}{\hat{z}_{j-1}-z_{j-1}}\right)\gamma_{|j-2|}.$$

Define $\alpha_{|j|}$ to be $(\hat{z}_j-z_j)/(\hat{z}_{j-1}-z_{j-1})$ when the denominator is not 0. If $\hat{z}_{-1}=z_{-1}$ in addition to $\hat{z}_0=z_0$, then $z_j=\hat{z}_j$ for all j. Since these sequences are supposed to be distinct, $\hat{z}_{-1}\neq z_{-1}$. Then α_0 exists and it is 0. Let $k=|j|$. If both α_k and α_{k+1} exist, then the preceding equation becomes

$$\alpha_k=1+\frac{\gamma_{k+2}}{\alpha_{k+1}}.$$

If α_k exists but $\hat{z}_{-k-2}=z_{-k-2}$ then $\alpha_k=1$.

We now consider several cases. First, suppose there is $m<0$ so that $\hat{z}_m=z_m$. We have seen that $m\leqslant -2$. If $\hat{z}_{-2}=z_{-2}$, the recurrence shows that $\hat{z}_{-1}=z_{-1}$ which has already been forbidden. Therefore $m\leqslant -3$, and we may assume $\hat{z}_j\neq z_j$ for $m<j<0$. Then $\alpha_k=1+(\gamma_{k+2}/\alpha_{k+1})$ for $0\leqslant k<|m|-2$, $\alpha_{|m|-2}=1$, and $\alpha_0=0$. Combining the equations results in

$$\alpha_0=0=[[\gamma_2,\gamma_3,\ldots,\gamma_{|m|-1}]]$$

which is impossible according to Theorem 8.2 since $\gamma_k\in D$ for $k\geqslant 1$.

Now assume $\hat{z}_m\neq z_m$ for all $m<0$. Every $\alpha_{|m|}$ is then defined, and

$$\alpha_0=0=\left[\left[\gamma_2,\gamma_3,\ldots,\gamma_k,\frac{\gamma_{k+1}}{\alpha_k}\right]\right].$$

If there is some k for which $|\gamma_{k+1}/\alpha_k|\leqslant\frac{1}{4}$, then (since all z's with $|z|\leqslant\frac{1}{4}$ are in D as are all γ_j's) we have produced a finite continued fraction equal to 0 with all entries in D. Again, this is impossible by Theorem 8.2.

So in addition to $\hat{z}_m \neq z_m$ for all $m<0$ we may assume $4\,|\gamma_{k+1}| > |\alpha_k|$ for all $k>0$. If we let $\delta_k = |\hat{z}_{-k} - z_{-k}| \neq 0$ for $k>0$ and recall the definition of α_k, the inequality becomes

$$4\,|\gamma_{k+1}|\,\delta_{k+1} > \delta_k.$$

We analyze the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ with manipulations similar to what was done in the proof of Theorem 4.2.

$$\begin{aligned}
\delta_{k+2} &= |\hat{z}_{-k-2} - z_{-k-2}| = \left|\sqrt{\hat{z}_{-k} - \hat{z}_{-k-1}} - \sqrt{z_{-k} - z_{-k-1}}\right| \\
&= \frac{|(\hat{z}_{-k} - \hat{z}_{-k-1}) - (z_{-k} - z_{-k-1})|}{\left|\sqrt{\hat{z}_{-k} - \hat{z}_{-k-1}} + \sqrt{z_{-k} - z_{-k-1}}\right|} \\
&\geqslant \frac{|\hat{z}_{-k} - z_{-k-1}| - |\hat{z}_{-k} - z_{-k-1}|}{\left|\sqrt{\hat{z}_{-k} - \hat{z}_{-k-1}} + \sqrt{z_{-k} - z_{-k-1}}\right|} \\
&\geqslant \frac{|\hat{z}_{-k} - z_{-k-1}|\,(1 - 4\,|\gamma_{k+1}|)}{\left|\sqrt{\hat{z}_{-k} - \hat{z}_{-k-1}} + \sqrt{z_{-k} - z_{-k-1}}\right|} \\
&\geqslant \delta_{k+1} \frac{(1 - 4\,|\gamma_{k+1}|)}{\left|\sqrt{\hat{z}_{-k} - \hat{z}_{-k-1}} + \sqrt{z_{-k} - z_{-k-1}}\right|}.
\end{aligned}$$

Several comments should be made about the algebra. Since we are considering *AIR* sequences with entries in $\overline{Q_1}\backslash\{0\}$, the square roots are well-defined and non-zero. The denominators are sums of elements of $\overline{Q_1}\backslash\{0\}$ and so cannot be 0.

The sequences $\{\gamma_k\}$ and $\{|\sqrt{\hat{z}_{-k} - \hat{z}_{-k-1}} + \sqrt{z_{-k} - z_{-k-1}}|\}$ both have limit 0 as $k \to \infty$. From this and the inequalities above we can conclude that there is $K>1$ and $k_* > 0$ so that for $k \geqslant k_*$, $\delta_{k+2} \geqslant K\delta_{k+1}$. But each δ_k is positive which implies that the limit of $\{\delta_k\}$ as $k \to \infty$ is ∞. Since the limit of the sequence $\{\hat{z}_{-k} - z_{-k}\}$ is 0 by Proposition 8.1, we again have a contradiction. ∎

9. NON-ANALYTICITY OF THE FUNCTIONAL EQUATION'S SOLUTION NEAR 0

The function L introduced in Section 3 satisfies the functional equation

$$x^2 + L(x) = L(L(x))$$

for all $x \geqslant 0$. We have described how to extend this function to h_1 which is holomorphic on $\mathbb{C}_-$. Here we investigate the behavior of L near 0.

Some formal power series solutions of this equation can be created if the initial condition $L(0)=0$ is given. Suppose $L(x)$ is written as a formal Taylor series $\sum_{n=0}^{\infty}(L^{(n)}(0)/n!)\,x^n$. We differentiate the functional equation and get

$$2x+L'(x)=L'(L(x))\,L'(x).$$

The initial condition implies that $L'(0)$ is 0 or 1. Further differentiation reveals that if $L(0)=0$ and $L'(0)=0$, then all coefficients are 0. There is, however, a unique non-zero formal power series solution with integer coefficients to the functional equation when $L'(0)=1$. Here is the beginning of this solution:

$$\begin{aligned}&x+x^2-2x^3+9x^4-56x^5+420x^6-3572x^7+33328x^8-3\;34354x^9\\&\quad+35\;59310x^{10}-398\;38760x^{11}+4657\;43720x^{12}-56589\;83108x^{13}\\&\quad+7\;11919\;48512x^{14}+\cdots\end{aligned}$$

The observed almost geometric growth of the series coefficients suggests that L can be extended holomorphically to a neighborhood of 0. We prove that this is *not* correct. The proof initially considers h_{-1}, whose behavior near 0 seems easier to analyze than that of $h_1=L$. If h_{-1} were holomorphic near 0, its power series at 0 would be the formal inverse of the one displayed above and have real rational coefficients.

We begin with some preliminary lemmas. Contrast the first with Lemma 6.1, which discusses the size of descendants rather than ancestors in sequences satisfying the QF recurrence.

LEMMA 9.1. *Suppose* $\{z_j\}_{j\leqslant 0}$ *is a complex sequence and* $z_j^2+z_{j+1}=z_{j+2}$ *for* $j\leqslant -2$. *If* $R=\max(2,|z_{-1}|,|z_0|)$, *then* $|z_j|\leqslant R$ *for all* $j\leqslant 0$.

Proof. If the lemma is false, there are some z_j's with $|z_j|>R$. Let $m=\max\{j<0:|z_j|>R\}$. Certainly $m\leqslant -2$ and $|z_j|\leqslant R$ for $m<j\leqslant 0$. Since $z_m^2+z_{m+1}=z_{m+2}$ we know that $|z_m^2+z_{m+1}|=|z_{m+2}|\leqslant R$. But

$$|z_m^2+z_{m+1}|\geqslant |z_m^2|-|z_{m+1}|>R^2-R=R(R-1)\geqslant R$$

which is a contradiction. ∎

The following result would follow easily from Lemma 8.3 if we knew that all the z_k's and z were in $\overline{Q_1}\backslash\{0\}$. The result is needed, however, when z is a negative real number in an effort to investigate candidates for $h_{-1}(z)$.

LEMMA 9.2. *Suppose $z \in \mathbb{R}$ and $\{z_k\}_{k \in \mathbb{N}}$ is a sequence in $\mathscr{H}_+$ with $\lim_{k \to \infty} z_k = z$. For each $k \geqslant 1$, let $\{z_{k,j}\}_{j \in \mathbb{Z}}$ be an AIR sequence through z_k. Then $\{z_{k,-1}\}_{k \in \mathbb{N}}$ is bounded.*

Proof. If the lemma were false, we could assume after passing to a subsequence that we have $z \in \mathbb{R}$, a sequence $\{z_k\}_{k \in \mathbb{N}}$ in $\mathscr{H}_+$, and, for each $k \geqslant 1$, an *AIR* sequence $\{z_{k,j}\}_{j \in \mathbb{Z}}$ through z_k with $|z_{k,-1}| \to \infty$.

Let $\alpha_k = |z_{k,-1}|$ and $\theta_{k,j} = \arg(z_{k,j})$ for $k \in \mathbb{N}$ and $j \leqslant 0$. We investigate the asymptotic behavior of $\theta_{k,j}$ for $j \in \{-1, -2, -3, -4\}$ as $k \to \infty$ and arrive at a contradiction. Define δ_k by $(\pi - \theta_{k,-1}) + 2\theta_{k,-2} + \delta_k = 2\pi$ so that $2\theta_{k,-2} - \theta_{k,-1} = \pi - \delta_k$ (as in Fig. 7). The law of cosines gives

$$\alpha_k^2 + |z_{k,-2}|^4 - 2\alpha_k\, |z_{k,-2}|^2 \cos(\delta_k) = |z_k|^2.$$

Since $z_k \to z$ and $|\alpha_k| \to \infty$ we see that $|z_{k,-2}| \to \infty$ and $\cos(\delta_k)$ must eventually be positive. Complete the square:

$$(\alpha_k - \cos(\delta_k)\, |z_{k,-2}|^2)^2 + |z_{k,-2}|^4\, (1 - (\cos(\delta_k))^2) = |z_k|^2.$$

Then $(1 - (\cos(\delta_k))^2) \leqslant |z_k|^2 / |z_{k,-2}|^4$ so $1 - \cos(\delta_k)^2 \to 0$ as $k \to \infty$. Since $\delta_k \in [0, \pi]$ and, as noted, $\cos(\delta_k)$ is eventually positive, $\delta_k \to 0$ as

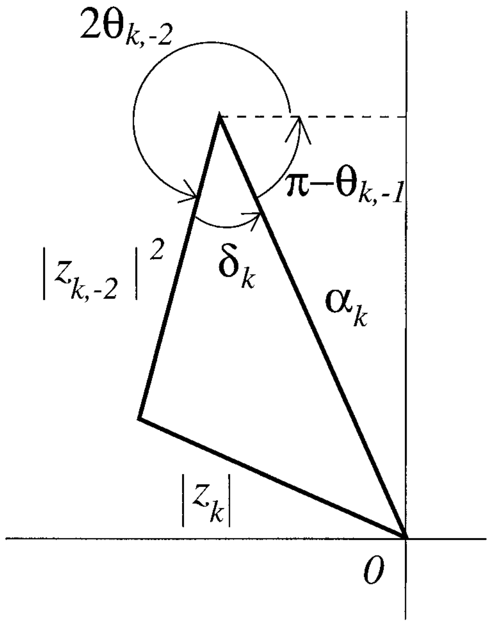

FIG. 7. The law of cosines again.

$k \to \infty$. So $2\theta_{k,-2} - \theta_{k,-1} = \pi - \delta_k \to \pi$ as $k \to \infty$. But $2\theta_{k,-2} - \theta_{k,-1} = \theta_{k,-2} + (\theta_{k,-2} - \theta_{k,-1}) \leqslant \theta_{k,-2} < \pi$, and $\theta_{k,-2} \to \pi$. Also, $\theta_{k,-1} \to \pi$ because $\theta_{k,-2} \leqslant \theta_{k,-1} < \pi$. Since $e^{i\pi} = -1$, we know

$$\lim_{k \to \infty} \frac{z_{k,-1}}{\alpha_k} = -1.$$

The recurrence relation gives

$$\left(\frac{z_{k,-2}}{\sqrt{\alpha_k}}\right)^2 + \left(\frac{z_{k,-1}}{\alpha_k}\right) = \frac{z_k}{\alpha_k}$$

which yields $\lim_{k \to \infty} (z_{k,-2}/\sqrt{\alpha_k})^2 = -\lim_{k \to \infty} (z_{k,-1}/\alpha_k) = 1$. But $\theta_{k,-2} \to \pi$ so that $\arg(z_{k,-2}/\sqrt{\alpha_k}) \to \pi$. Therefore

$$\lim_{k \to \infty} \frac{z_{k,-2}}{\sqrt{\alpha_k}} = -1.$$

Go backwards another step with the recurrence relation

$$\left(\frac{z_{k,-3}}{\sqrt{\alpha_k}}\right)^2 + \frac{1}{\sqrt{\alpha_k}} \left(\frac{z_{k,-2}}{\sqrt{\alpha_k}}\right) = \frac{z_{k,-1}}{\alpha_k}$$

and see that

$$\lim_{k \to \infty} \left(\frac{z_{k,-3}}{\sqrt{\alpha_k}}\right)^2 = -1.$$

Since $z_{k,-3} \in \mathscr{H}_+$ for all $k \in \mathbb{N}$,

$$\lim_{k \to \infty} \frac{z_{k,-3}}{\sqrt{\alpha_k}} = i = \exp\left(i \frac{\pi}{2}\right).$$

A final backwards recurrence step gives

$$\left(\frac{z_{k,-4}}{\alpha_k{}^{1/4}}\right)^2 + \left(\frac{z_{k,-3}}{\sqrt{\alpha_k}}\right) = \frac{z_{k,-2}}{\sqrt{\alpha_k}}$$

and thus

$$\lim_{k \to \infty} \left(\frac{z_{k,-4}}{\alpha_k{}^{1/4}}\right)^2 = -1 - i.$$

The choice of square root is clear because again all numbers are in $\mathscr{H}_+$. Therefore

$$\lim_{k\to\infty} \frac{z_{k,-4}}{\alpha_k^{1/4}} = 2^{1/4} \exp\left(i\frac{5\pi}{8}\right).$$

But $\frac{5}{8} > \frac{1}{2}$ so that eventually we cannot have $\arg(z_{k,-4}) < \arg(z_{k,-3})$, a contradiction since $\{z_{k,j}\}_{j\in\mathbb{Z}}$ is supposed to be an *AIR* sequence for every $k \in \mathbb{N}$. ∎

We apply the two earlier lemmas to get a local uniform bound on all iterates of h_{-1}.

LEMMA 9.3. *If $r > 0$, let $\mathscr{H}_{+,r} = \{z \in \mathscr{H}_+ : |z| < r\}$. Then there is a constant $M = M(r)$ so that $|h_{-1}^{[k]}(z)| \leqslant M(r)$ for all $k \in \mathbb{N}$ and all $z \in \mathscr{H}_{+,r}$.*

Proof. Given $z \in \mathscr{H}_+$, Theorem 8.1 asserts that there exists an *AIR* sequence through z, and that such a sequence has $z_k = h_{-1}^{|k|}(z)$ for k negative. But Lemma 9.1 allows us to bound the values of $h_{-1}^{|k|}(z)$ for all $z \in \mathscr{H}_{+,r}$ if we can find a bound for $h_{-1}(z)$.

If there is no bound, there is a sequence $\{\zeta_j\}_{j\in\mathbb{N}}$ in $\mathscr{H}_{+,r}$ so that $\lim_{j\to\infty} |h_{-1}(\zeta_j)| = \infty$. An appropriate subsequence of this bounded sequence will converge to $\zeta \in \mathscr{H}_+ \cup \mathbb{R}$. The alternative $\zeta \in \mathscr{H}_+$ is impossible, since the domain of the continuous function h_{-1} includes ζ. But $\zeta \in \mathbb{R}$ is also ruled out by the previous lemma. So no such sequence exists and this result is true. ∎

The lemmas are used to construct a candidate for an *AIR* sequence through a negative real number. These sequences are a key ingredient of the proof that h_{-1} cannot be holomorphic in a neighborhood of 0.

THEOREM 9.1. *If $z \in \mathbb{R}_{<0}$, there is a bounded sequence $\{\zeta_j\}_{j\leqslant 0}$ in $\overline{\mathscr{H}_+}\setminus\{0\}$ so that*

(1) $0 < \arg \zeta_j \leqslant \arg \zeta_{j+1} \leqslant 2 \arg \zeta_j$ *for all* $j < 0$.

(2) $\zeta_{j-2}^2 + \zeta_{j-1} = \zeta_j$ *for all* $j \leqslant 0$.

(3) $\zeta_0 = z$.

Proof. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathscr{H}_+$ whose limit is z. For each $j \in \mathbb{Z}$, let $z_{k,j} = h_{-1}^{[j]}(z_k)$ when $j < 0$ and let $z_{k,0} = z_k$. When $j > 0$, let $z_{k,j} = z_{k,j-1} + z_{k,j-2}^2$. Then $\{z_{k,j}\}_{j\in\mathbb{Z}}$ is an *AIR* sequence through z_k by Theorem 8.1. Lemma 9.3 applies since the z_k's must be bounded because they converge, so there is $M > 0$ independent of $j \leqslant 0$ and $k \in \mathbb{N}$ with

$$|z_{k,j}| \leqslant M.$$

Then take subsequences and diagonalize so that $\lim_{k\to\infty} z_{k,j} = \zeta_j \in \overline{\mathscr{H}_+}$ exists for each $j \leqslant 0$. Continuity implies that $\zeta_0 = z$, $\lim_{k\to\infty} z_{k,j} = \zeta_j \in \overline{\mathscr{H}_+}$ exists for $j > 0$, and $\{\zeta_j\}_{j\in\mathbb{Z}}$ satisfies the recurrence: $\zeta_j = \zeta_{j-1} + \zeta_{j-2}^2$ for all $j \in \mathbb{Z}$. Also $0 \leqslant \arg\zeta_j \leqslant \arg\zeta_{j-1} \leqslant 2\arg\zeta_j$ for $j \leqslant 0$ when ζ_j and ζ_{j-1} are non-zero.

If there is $j < 0$ with $\zeta_j \neq 0$ and $\arg\zeta_j = 0$, the preceding inequality implies that for $k \leqslant j$, either ζ_k is 0 or $\arg\zeta_k = 0$. Thus all such ζ_k's are real and non-negative, which implies that every ζ_k is in $\mathbb{R}_{\geqslant 0}$. But $\zeta_0 = z < 0$. Therefore $\arg\zeta_j > 0$ for any $j < 0$ with $\zeta_j \neq 0$.

In fact, $\zeta_j \neq 0$ for all $j \leqslant 0$. If this is false, let $J = \max\{ j \in \mathbb{Z} : j < 0 \text{ and } \zeta_j = 0\}$. The recurrence relation becomes

$$\zeta_{J-1}^2 + \zeta_J = \zeta_{J-1}^2 = \zeta_{J+1}$$

which shows that $\zeta_{J-1} \neq 0$ and $0 < \arg\zeta_{J-1} \leqslant \frac{\pi}{2}$. The last inequality is a consequence of $2\arg\zeta_{J-1} = \arg\zeta_{J+1} \leqslant \pi$ since $\zeta_{J+1} \in \overline{\mathscr{H}_+}$. Going backwards one more time in the recurrence relation gives

$$\zeta_{J-2}^2 + \zeta_{J-1} = \zeta_J = 0$$

so that ζ_{J-2} is also non-zero. We know $0 \leqslant \arg\zeta_{J-2} \leqslant \arg\zeta_{J-1} \leqslant \frac{\pi}{2}$ and $\arg\zeta_{J-2} \leqslant 2\arg\zeta_{J-1}$. Lemma 8.3 then implies $|z_J| \geqslant |z_{J-1}| > 0$ which is a contradiction. ∎

At least one ζ_j whose existence we have proven above for $j < 0$ must be in $\mathscr{H}_+$ rather than $\mathbb{R}_{<0}$. For suppose all such ζ_j's are negative real numbers. Since $\zeta_{j-2}^2 + \zeta_{j-1} = \zeta_j$, we know $\zeta_{j-1} < \zeta_j < 0$. Also the set $\{\zeta_j\}_{j\leqslant 0}$ is bounded, implying that $\lim_{j\to-\infty}\zeta_j = \zeta$ exists. But $\zeta^2 + \zeta = \zeta$ so that ζ must be 0, contradicting $\zeta \leqslant \zeta_j < 0$.

THEOREM 9.2. *h_{-1} has no holomorphic extension to any neighborhood of* 0.

Proof. Suppose h_{-1} had a holomorphic extension to $D_r = \{z \in \mathbb{C} : |z| < r\}$. The formal power series for h_{-1} shows that $h_{-1}(0) = 0$, $h'_{-1}(0) = 1$, and $h_{-1}(\bar{z}) = \overline{h_{-1}(z)}$ so $h_{-1}(D_r \cap \mathbb{R}) \subset \mathbb{R}$.

Select $z \in \mathbb{R}_{<0} \cap D_r$ so that $h_{-1}(z) \in \mathbb{R}_{<0} \cap D_r$ and select $\{z_k\}_{k\in\mathbb{N}}$, a sequence in $\mathscr{H}_+$ with $\lim_{k\to\infty} z_k = z$. As in the proof of the previous theorem, we may assume by taking subsequences and diagonalizing that $\lim_{k\to\infty} h_{-1}^{[|j|]}(z_k) = \zeta_j$ exists for all $j < 0$. We know $\zeta_{-1} = h_{-1}(z) \in \mathbb{R}_{<0}$. The remarks preceding the statement of this theorem imply that *some* ζ_j with $j < 0$ must be in $\mathscr{H}_+$. Let $J = \max\{ j \in \mathbb{Z} : \zeta_j \in \mathscr{H}_+\}$. $J < -1$ and $\zeta_j \in \mathbb{R}$ for $J < j \leqslant 0$. The previous theorem implies that such ζ_j's are actually in $\mathbb{R}_{<0}$.

Certainly $\zeta_J^2 = \zeta_{J+2} - \zeta_{J+1}$. If $\zeta_{J+2} \geqslant \zeta_{J+1}$, then $\zeta_J \in \mathbb{R}$ which is a contradiction. Therefore $\zeta_{J+2} < \zeta_{J+1}$.

Let $a = -\zeta_{J+1}$ and $b = -\zeta_{J+2}$. Then a and b are real and $0 < a < b$. Since $\zeta_J \in \overline{\mathscr{H}_+}$, $\zeta_J = \sqrt{b-a}\exp(i\frac{\pi}{2})$. We go backwards once more. The equation

$$\zeta_{J-1}^2 + \zeta_J = \zeta_{J+1} = -a$$

shows that $\zeta_{J-1}^2 = -a - \sqrt{b-a}\exp(i\frac{\pi}{2}) = \exp(i\pi)(a + \sqrt{b-a}\exp(i\frac{\pi}{2}))$. Write $a + \sqrt{b-a}\exp(i\frac{\pi}{2})$ as $\rho\exp(i\psi)$ with $\rho > 0$ and $0 < \psi < \frac{\pi}{2}$. Because $\zeta_{J-1} \in \overline{\mathscr{H}_+}$ also, we may write $\zeta_{J-1} = \sqrt{\rho}\exp(i(\frac{\pi}{2} + \frac{\psi}{2}))$. Thus we have

$$\arg\zeta_{J-1} = \frac{\pi}{2} + \frac{\psi}{2} > \frac{\pi}{2} = \arg\zeta_J$$

contradicting the argument increasing behavior of the sequence $\{\zeta_j\}_{j \leqslant 0}$: $0 < \arg\zeta_{j-1} \leqslant \arg\zeta_j \leqslant \pi$ for $j \leqslant 0$. ∎

We know that $L = h_1$ and h_{-1} are inverse functions on $\mathbb{R}_{\geqslant 0}$ and that $h_1'(0) = 1$. A holomorphic extension of h_1 to a neighborhood of 0 would thus imply a similar extension for h_{-1}.

Corollary 9.1. *$L = h_1$ has no holomorphic extension to any open neighborhood of* 0 *so L cannot be extended to a real analytic function in any open neighborhood of* 0.

Eugene R. Speer has investigated h_1 numerically using the formal power series combined with the functional equation. The behavior of iterates of h_1 seems to be quite complicated. Note that h_{-1} cannot be extended even continuously over any point of $\mathbb{R}_{<0}$. If h_{-1} were extended continuously to $\mathbb{C}_- \cup \{z\}$ with $z \in \mathbb{R}_{<0}$, the conjugate symmetry and injectivity of h_{-1} force $h_{-1}(z)$ to be in $\mathbb{R}_{<0}$. Arguments similar to those in the previous two theorems show this is impossible.

Although L is not analytic near 0, we will show that it is C^∞ in $[0, \infty)$. Of course, there are classical examples (e.g., $\sum_{n=0}^{\infty} \frac{1}{n!} z^{2^n}$) of power series whose radius of convergence is 1 and whose boundary values on the circle of convergence are C^∞ and nowhere real analytic.

10. SMOOTHNESS OF THE FUNCTIONAL EQUATION'S SOLUTION NEAR 0

The original functional equation can be rewritten as

$$L(w) = w + (L^{[-1]}(w))^2 \tag{$*$}$$

which immediately suggests using a contraction mapping on an appropriate set of functions. Good properties of L could then be proved by writing it as a limit of some approximation scheme in that set. We follow this outline in part. For each positive integer m, the set of functions will be a subset of $C^m[0, r]$ for some $r > 0$. The specification of the subset and the approximation method take some effort. We set the stage by giving another proof that $(*)$ has a unique continuous strictly increasing solution, and hope the more elaborate framework used to prove smoothness at 0 will then be easier to understand.

For $r > 0$, let $X_r = C_{\mathbb{R}}([0, r]) = C^0([0, r])$, the Banach space of continuous real-valued functions on $[0, r]$ with the sup norm. X_r is also partially ordered by requiring that $f \leqslant g$ in X_r if $f(x) \leqslant g(x)$ for all $x \in [0, R]$. If L is a solution in X_r of $(*)$, then certainly L is strictly increasing and $L(x) \geqslant x$. $L^{[-1]}$ would also be increasing, so that $L(y) - L(x) \geqslant y - x$ for all x, y in $[0, r]$ with $x \leqslant y$. Of course this encodes the fact that L', if it exists, should be at least 1. We define a suitable subset of X_r:

$$G_r = \{ f \in X_r : f(0) = 0 \text{ and } f(y) - f(x) \geqslant y - x$$
$$\text{for all } x,\ y \in [0, r] \text{ with } 0 \leqslant x \leqslant y \}.$$

G_r is a closed, convex set in X_r. If $f \in G_r$, then f is strictly increasing, $f(x) \geqslant x$ for all $x \in [0, r]$, and the increasing function $f^{[-1]}$ maps $[0, f(r)]$ to $[0, r]$. Since $f(r) \geqslant r$, $f^{[-1]}$ may be restricted to $[0, r]$ and is continuous there. This restriction will be called $\Gamma_r(f)$. The reflection of the property $f(x) \geqslant x$ is $\Gamma_r(f)(x) \leqslant x$ so that

$$f(\Gamma_r(f)(y)) - f(\Gamma_r(f)(x)) = y - x \geqslant \Gamma_r(f)(y) - \Gamma_r(f)(x).$$

If $f \in G_r$, define a map $F_r \colon G_r \to X_r$ by

$$F_r(f)(x) = x + (\Gamma_r(f)(x))^2.$$

Solving $(*)$ is equivalent to finding a fixed point of F_r.

LEMMA 10.1. *$F_r \colon G_r \to X_r$ is continuous, and $F_r(G_r) \subset G_r$, and $\overline{F_r(G_r)}$ is compact. F_r is order-reversing: if $f \leqslant g$, then $F_r(g) \leqslant F_r(g)$.*

Proof. If $f \in G_r$ and $g = \Gamma_r(f)$, then $g(0) = 0$, g is increasing, and $g(x) \leqslant x$. Thus $0 \leqslant F_r(f)(x) = (g(x))^2 + x \leqslant r^2 + r$ for $x \in [0, r]$. Also, $F_r(f)(0) = 0$ and F_r is the sum of two increasing functions and is therefore increasing. Since $(g(x))^2$ is increasing, if x, $y \in [0, r]$ with $x \leqslant y$,

$$F_r(f)(y) - F_r(f)(x) = y - x + (g(y))^2 - (g(x))^2 \geqslant y - x$$

and we have shown that $F_r(G_r) \subset G_r$ with the uniform bound $\|F_r(f)\| \leqslant r^2 + r$. That the closure of the uniformly bounded set $F_r(G_r)$ is compact follows if we prove equicontinuity.

We first suppose that f and $\tilde{f}$ are in G_r with $g = \Gamma_r(f)$ and $\tilde{g} = \Gamma_r(\tilde{f})$. Then

$$\begin{aligned}|g(x) - \tilde{g}(x)| &\leqslant |f(g(x)) - f(\tilde{g}(x))| = |x - f(\tilde{g}(x))| \\ &= |\tilde{f}(\tilde{g}(x)) - f(\tilde{g}(x))| \leqslant \|f - \tilde{f}\|.\end{aligned}$$

Thus

$$\begin{aligned}|F_r(f)(x) - F_r(\tilde{f})(x)| &= |(g(x))^2 - (\tilde{g}(x))^2| \\ &\leqslant |g(x) + \tilde{g}(x)|\,|g(x) - \tilde{g}(x)| \leqslant (2r)\,\|f - \tilde{f}\|\end{aligned}$$

showing that F_r is a Lipschitz map with Lipschitz constant at most $2r$.

Now consider $F_r(f)(y) - F_r(f)(x)$ when $0 \leqslant x \leqslant y \leqslant r$.

$$\begin{aligned}0 \leqslant F_r(f)(y) - F_r(f)(x) &= (y - x) + (g(y))^2 - (g(x))^2 \\ &= (y - x) + (g(y) - g(x))(g(y) + g(x)) \\ &\leqslant (y - x) + (y - x)(g(y) + g(x)) \leqslant (1 + 2r)(y - x)\end{aligned}$$

so each $F_r(f)$ is a Lipschitz function with Lipschitz constant at most $1 + 2r$. The family of functions $F_r(G_r)$ is certainly equicontinuous.

We now check the stated order-reversing property. Again, assume that f and $\tilde{f}$ are in G_r with $g = \Gamma_r(f)$ and $\tilde{g} = \Gamma_r(\tilde{f})$, and, additionally, $f \leqslant \tilde{f}$. Thus $f(x) \leqslant \tilde{f}(x)$, so that $\tilde{g}(f(x)) \leqslant \tilde{g}(\tilde{f}(x)) \leqslant x = g(f(x))$ for all x's such that $f(x) \in [0, r]$. Every element of $[0, r]$ can be so written since $f(x) \geqslant x$. We have verified $\tilde{g} \leqslant g$. ∎

We build a solution to $(*)$ by changing r, the right-hand end point. Suppose that $R > r > 0$. Define the restriction mapping $\mathscr{R}_{R,r}\colon X_R \to X_r$ by $(\mathscr{R}_{R,r}(f))(x) = f(x)$ when $x \in [0, r]$. Then $\mathscr{R}_{R,r}(F_R(f)) = F_r(\mathscr{R}_{R,r}(f))$ for $f \in X_R$. If f is a solution of $(*)$ in X_R (so $F_R(f) = f$) it follows that $\mathscr{R}_{R,r}(f)$ is a solution of $(*)$ in X_r. If $f \in C[0, \infty)$, then $\mathscr{R}_{\infty,r}(f)$ is the restriction of f to $[0, r]$, an element of X_r.

Lemma 10.2. *If $r > 0$, there is a unique $L_r \in G_r$ with $F_r(L_r) = L_r$. If $R > r$, then $\mathscr{R}_{R,r}(L_R) = L_r$, and there is a unique continuous strictly increasing function L solving $(*)$ on $[0, \infty)$.*

Proof. G_r is a closed, convex set, and by the previous lemma, $F_r\colon G_r \to G_r$ is a continuous mapping whose range is contained in a compact subset of G_r. The Schauder Fixed Point Theorem (see [5] or [13]) applies and F_r must have a fixed point. If f_r is any such fixed point in G_r and if $x \in [0, r]$, then the sequence $\{x_j\}_{j \leqslant 0}$ defined by $x_j = f_r^{[j]}(x)$ (the iterates of the inverse of the one-one map f_r applied to x) must satisfy the QF recurrence. Given x, Theorem 3.2 declares that there is exactly one x_{-1} allowing a sequence to be continued "backwards" indefinitely in $\mathbb{R}_{>0}$ (the theorem applies to doubly infinite sequences, but the recurrence can always be pushed forward in $\mathbb{R}_{>0}$). Therefore $f_r(x)$ and $f_r^{[-1]}(x)$ are uniquely determined for any $x \in [0, r]$, and F_r acting on G_r has a unique fixed point, L_r. Uniqueness also confirms that $\mathscr{R}_{R,r}(L_R) = L_r$ when $R > r$, and allows the definition of $L(x)$: it is $L_r(x)$ for any $r \geqslant x \geqslant 0$. The uniqueness of L also follows from uniqueness of the L_r's. ∎

The previous lemmas can be used to "construct" a solution to $(*)$. Define the sequence $\{\mathscr{L}_n\}_{n \in \mathbb{N}}$ in G_r by $\mathscr{L}_0(x) = x$ for all $x \in [0, r]$ and $\mathscr{L}_{n+1} = F_r(\mathscr{L}_n)$ for $n \geqslant 0$. These functions are all in $C[0, \infty)$, and are in G_r by restriction. We use two facts. First, $\mathscr{L}_0$ is minimal in G_r: $\mathscr{L}_0 \leqslant f$ for all $f \in G_r$. And, second, even compositions of F_r (that is, the maps $F_r^{[2n]}$) are order-preserving, and odd compositions of F_r (the maps $F_r^{[2n+1]}$) are order-reversing. We must have

$$\mathscr{L}_0 \leqslant \mathscr{L}_2 \leqslant \cdots \leqslant \mathscr{L}_{2n} \leqslant \mathscr{L}_{2n+2} \leqslant \cdots \leqslant \mathscr{L}_{2n+3} \leqslant \mathscr{L}_{2n+1} \leqslant \cdots \leqslant \mathscr{L}_3 \leqslant \mathscr{L}_1.$$

Also, if $f \in G_r$ and $n \in \mathbb{N}$, then $\mathscr{L}_{2n} \leqslant F_r^{[2n]}(f)$ and $F_r^{[2n+1]}(f) \leqslant \mathscr{L}_{2n+1}$. L is a fixed point of F_r and thus must be between all $\mathscr{L}_{2n}$'s and $\mathscr{L}_{2n+1}$'s.

Suppose that $\mathscr{L}_e$ is the pointwise limit of $\{\mathscr{L}_{2n}\}$ on $[0, \infty)$ and that $\mathscr{L}_o$ is the pointwise limit of $\{\mathscr{L}_{2n+1}\}$ on $[0, \infty)$. The equicontinuity previously shown verifies that $\mathscr{L}_{2n} \to \mathscr{L}_e$ and $\mathscr{L}_{2n+1} \to \mathscr{L}_o$ uniformly in $C^0([0, r])$ for fixed $r > 0$. We will show that $\mathscr{L}_e$ and $\mathscr{L}_o$ agree on all of $[0, \infty)$. We know that F_r is a Lipschitz map with Lipschitz constant at most $2r$. Take $r \in (0, \frac{1}{2})$. Then

$$|\mathscr{L}_{2k}(x) - \mathscr{L}_{2k+1}(x)| \leqslant (2r)^{2k} \, \|\mathscr{L}_0 - \mathscr{L}_1\|$$

for any $x \in [0, r]$ and $k \in \mathbb{N}$. $\mathscr{L}_o$ and $\mathscr{L}_e$ must agree on $[0, r]$ because $(2r)^{2k} \to 0$ as $k \to \infty$.

For any $s > 0$, both $\mathscr{L}_e$ and $\mathscr{L}_o$ are fixed points of $F_s^{[2]}$ in $C[0, s]$. This translates to

$$\mathscr{L}_e(\mathscr{L}_e(x) + x^2) = \mathscr{L}_e(x) + x^2 + (\mathscr{L}_e(x))^2 \qquad \text{and}$$

$$\mathscr{L}_o(\mathscr{L}_o(x) + x^2) = \mathscr{L}_o(x) + x^2 + (\mathscr{L}_o(x))^2$$

for all $x > 0$. Since $\mathscr{L}_o(x) \geqslant x$ and $\mathscr{L}_e(x) \geqslant x$, given $y \in [0, r + r^2]$ there is $x \in [0, r]$ with $\mathscr{L}_o(x) + x^2 = \mathscr{L}_e(x) + x^2 = y$. The equations above show that the functions $\mathscr{L}_o$ and $\mathscr{L}_e$ are equal at least on $[0, r+r^2]$. We may repeat this argument to show that the agreement of $\mathscr{L}_e$ and $\mathscr{L}_o$ extends to all of $[0, \infty)$.

THEOREM 10.1. *There is a unique strictly increasing continuous map* $L: [0, \infty) \to [0, \infty)$ *such that* $L(w) = w + (L^{[-1]}(w))^2$ *for all* $w \in [0, \infty)$. *For every* $r > 0$, *the unique solution* L_r *in* G_r *of the equation* $F_r(f) = f$ *is* $\mathscr{R}_{\infty, r}(L)$. *If* $\mathscr{L}_0(x) = x$ *and* $\mathscr{L}_n = F_r^{[n]}(\mathscr{L}_0)$, *then* $\{\mathscr{L}_{2n}\}_{n \in \mathbb{N}}$ (*respectively,* $\{\mathscr{L}_{2n-1}\}_{n \in \mathbb{N}}$) *is an increasing* (*respectively, decreasing*) *sequence in* X_r, *both converging to* L_r. *If* $f \in G_r$, $F_r^{[2n]}(f) \geqslant \mathscr{L}_{2n}$ *and* $F_r^{[2n+1]}(f) \leqslant \mathscr{L}_{2n+1}$.

We omit the subscript from $L_r \in X_r$ since it is now known to be unique and to be the restriction of $L \in C[0, \infty)$.

Formulas for the first few $\mathscr{L}_n$'s can be used to estimate how rapidly L is approximated by $\mathscr{L}_n$ in X_r. Since $\mathscr{L}_0(x) = x$, we know $\mathscr{L}_1(x) = x + x^2$. The estimate $x \leqslant L(x) \leqslant x + x^2$ was already used to draw a graph of L, the figure following Proposition 4.1. Then $\mathscr{L}_1^{[-1]}(x) = \frac{1}{2}(\sqrt{1+4x} - 1)$, so that $\mathscr{L}_2(x) = x + (2x^2/(1 + 2x + \sqrt{1+4x}))$. We can continue one more step, after rewriting $\mathscr{L}_2(x)$ as $\frac{1}{2}((4x+1) - \sqrt{4x+1})$. Then $\mathscr{L}_2^{[-1]}(x) = \frac{1}{8}(4x + \sqrt{8x+1} - 1)$ which yields $\mathscr{L}_3(x) = x + ((8x^2 + 1 + (4x-1)\sqrt{8x+1})/32)$. We get the finer estimates

$$x + \frac{2x^2}{1 + 2x + \sqrt{1+4x}} \leqslant L(x) \leqslant x + \frac{8x^2 + 1 + (4x-1)\sqrt{8x+1}}{32}$$

valid for all $x \geqslant 0$.

Suppose N is an even positive integer and that $f \geqslant \mathscr{L}_N$ and $\tilde{f} \geqslant \mathscr{L}_N$. For example, we could ask that f and $\tilde{f}$ be elements of $F_r^{[N]}(G_r)$. If $g = \Gamma_r(f)$ and $\tilde{g} = \Gamma_r(\tilde{f})$, then as before

$$|F_r(f)(x) - F_r(\tilde{f})(x)| \leqslant \|f - \tilde{f}\|_{X_r} \, |g(x) + \tilde{g}(x)|.$$

Since $f \geqslant \mathscr{L}_N$, $F_r(f) \leqslant \mathscr{L}_{N+1}$. Therefore $g(x) = \Gamma_r(f)(x) \leqslant \Gamma_r(\mathscr{L}_N)(x)$. Since $\Gamma_r(f)$ and $\Gamma_r(\mathscr{L}_N)$ are increasing, we have proved

$$\|F_r(f) - F_r(\tilde{f})\|_{X_r} \leqslant \|f - \tilde{f}\|_{X_r} (2 \, \|\Gamma_r(\mathscr{L}_N)\|_{X_r}).$$

If $\leqslant$ is a partial order on a set S, let $\mathscr{I}_{A, B} = \{C \in S : A \leqslant C \leqslant B\}$. $\mathscr{I}_{A, \infty}$ is the set of C's in S satisfying $A \leqslant C$ only.

We see that F_r is a Lipschitz mapping from $\mathscr{I}_{\mathscr{L}_N, \infty}$ (respectively, $F_r^{[N]}(G_r)$) to itself with Lipschitz constant $2\, \|\Gamma_r(\mathscr{L}_N)\|_{X_r} = 2\Gamma_r(\mathscr{L}_N)(r)$. Note that $\Gamma_r(\mathscr{L}_N)(r)$ is a continuous strictly increasing function of r with $\Gamma_0(\mathscr{L}_N)(0) = 0$, so F_r is a contraction mapping after N iterations for small enough r's (any

$r < R$ where $\Gamma_R(\mathscr{L}_N)(R) = \frac{1}{2}$). For example, if $N = 2$, then $\Gamma_R(\mathscr{L}_2)(R) = \frac{1}{2}$ when $R = \mathscr{L}_2(\frac{1}{2}) = 3 - \sqrt{3}/2 \approx 0.66397$. Thus F_r is a contraction mapping after two iterations for r less than this R. When $N = 0$, the appropriate R is just $\frac{1}{2}$, so F_r is itself contracting when $r < \frac{1}{2}$.

Since L is fixed by F_r, the following result follows from standard facts about contraction mappings.

PROPOSITION 10.1. *If N is an even integer, then F_r is a Lipschitz map with Lipschitz constant $2\Gamma_r(\mathscr{L}_N)(r)$ on $\mathscr{I}_{\mathscr{L}_N, \infty}$ (respectively, $F_r^{[N]}(G_r)$). If $c = 2\Gamma_r(\mathscr{L}_N)(r) < 1$ then F_r is a contraction mapping on each of these sets, and has a unique fixed point. Also $\|\mathscr{L}_{n+N} - L\|_{X_r} \leqslant c^n \|\mathscr{L}_N - L\|_{X_r}$ for $n \geqslant 1$. For such r and n, and for any $x \in [0, r]$, $0 \leqslant L(x) - \mathscr{L}_{n+N}(x) \leqslant c^n \|\mathscr{L}_{N+1} - \mathscr{L}_N\|_{X_r}$ when n is even, and $0 \leqslant \mathscr{L}_{n+N}(x) - L(x) \leqslant c^{n-1} \|\mathscr{L}_{N+1} - \mathscr{L}_N\|_{X_r}$ when n is odd. For $N = 2$, any $r < (3 - \sqrt{3})/2$ suffices for these conclusions.*

Although this result only applies for small enough r, the equation $(*)$ allows computation of $L(x)$ for all x. For example, if L is known on $[0, r]$ for some $r > 0$, then $L^{[-1]}$ is known on $[0, L(r)]$ with $L(r) > r$. L's values on $[0, L(r)]$ are then given by $(*)$, etc., so that we can evaluate L on $[0, L^{[k]}(r)]$ for any positive integer k, and $\lim_{k \to \infty} L^{[k]}(r) = \infty$.

We now discuss the differentiability of L. Although we already know that f is real analytic on $(0, \infty)$ (Corollary 8.2), we prove independently that L is C^∞ on $[0, \infty)$. The method uses generalized measures of noncompactness, which we now briefly review.

Suppose (X, d) is a metric space, and $S \subset X$. Kuratowski [12] defined the *measure of noncompactness* $\alpha(S)$ of S by

$$\alpha(S) = \inf \left\{ \delta > 0 : S = \bigcup_{j=1}^{n} S_j,\ n < \infty \text{ and diameter}(S_j) \leqslant \delta \right\}.$$

If (X, d) is a complete metric space and $S \subset X$, $\bar{S}$ is compact exactly when $\alpha(S)$ is defined and $\alpha(S) = 0$. In what follows we shall assume $\alpha(S)$ is defined and finite when it is written (such S's are bounded).

If (X, d) is a complete metric space and $\{S_k\}_{k \in \mathbb{N}}$ is a decreasing sequence of closed, nonempty sets such that $\lim_{k \to \infty} \alpha(S_k) = 0$, Kuratowski proved that $S_\infty = \bigcap_{k \geqslant 1} S_k$ is compact and nonempty. Furthermore, if U is any neighborhood of S_∞, there is $N = N(U) \in \mathbb{N}$ so that $S_n \subset U$ for $n \geqslant N$.

If X is a Banach space whose metric is derived from a norm $\|\ \|$, G. Darbo [4] proved that Kuratowski's measure of noncompactness has nice additional properties. Suppose A and B are bounded subsets of X and λ is a scalar. Define $\lambda A = \{\lambda a : a \in A\}$ and $A + B = \{a + b : a \in A, b \in B\}$. $\overline{\text{co}}(A)$ will denote the closed, convex hull of A, the smallest closed, convex set containing A. Then Darbo proved

(a) $\alpha(\overline{\mathrm{co}}(A)) = \alpha(A)$.

(b) $\alpha(A+B) \leqslant \alpha(A) + \alpha(B)$.

(c) $\alpha(\lambda A) = |\lambda|\, \alpha(A)$.

(d) $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$.

The following remains true

(e) $\alpha(A) = 0$ if and only if $\bar{A}$ is compact.

Suppose that ω is a function which assigns a non-negative real number to each bounded subset A of a Banach space X. If ω satisfies properties (a)–(e) above then ω is called a *generalized measure of noncompactness*. Many definitions of these measures can be given but we use the one here. See [14–16] for further background on this material.

The following result is due to Darbo [4] who stated it for the Kuratowski measure of noncompactness. The same proof works for generalized measures of noncompactness.

THEOREM 10.2. *Suppose X is a Banach space, ω is a generalized measure of noncompactness, and G is a closed, bounded, convex set in X. If $F\colon G \to G$ is a continuous map and there is a non-negative constant $k < 1$ such that $\omega(F(A)) \leqslant k\omega(A)$ for all sets $A \subset G$, then F has a fixed point in G and $\{f \in G : F(f) = f\}$ is compact in G.*

We list some generalized measures of noncompactness to be used here. Suppose (S, d) is a compact metric space, and let $X = C(S)$, the continuous functions on S with the norm $\|f\| = \sup\{|f(s)|\colon s \in S\}$. If $\delta > 0$ and A is a bounded subset of X, define

$$\omega_\delta(A) = \sup\{|f(s) - f(t)|\colon f \in A,\ s, t \in S \text{ and } d(s, t) \leqslant \delta\}.$$

If $\omega(A) = \lim_{\delta \to 0^+} \omega_\delta(A)$, then ω is a generalized measure of noncompactness on X.

If $[a, b]$ is a closed, bounded interval in $\mathbb{R}$, $m \geqslant 1$ is an integer, and x_0 is a fixed point in $[a, b]$, then $C^m([a, b])$ will denote the m-times continuously differentiable functions on $[a, b]$ with the norm

$$\|f\| = \sum_{j=0}^{m-1} |f^{(j)}(x_0)| + \sup\{|f^{(m)}(x)| : a \leqslant x \leqslant b\}.$$

$C^m([a, b])$ is a Banach space with this norm. If A is a bounded subset of $C^m([a, b])$, define $\Omega_m(A)$ to be $\omega(\{f^{(m)}\colon f \in A\})$ where $\{f^{(m)}\colon f \in A\}$ is viewed as a subset of $C([a, b])$. Then Ω_m is a generalized measure of noncompactness on $C^m([a, b])$.

We begin by showing that f is C^1.

THEOREM 10.3. *Suppose L is the unique strictly increasing solution to* $(*)$. *Then* $L \in C^1([0, \infty))$.

Proof. Fix $r > 0$ which is less than $(3-\sqrt{3})/2$ (see Proposition 10.1). Recall that $\mathcal{I}_{\mathcal{L}_2, \mathcal{L}_1}$ is the "interval" of continuous functions on $[0, r]$ whose values are between the values of $\mathcal{L}_2$ and $\mathcal{L}_1$. Let $\mathcal{H}_r = C^1([0, r]) \cap \mathcal{I}_{\mathcal{L}_2, \mathcal{L}_1} \cap \{f \mid f' \geqslant 1\}$.Thus $f \in \mathcal{H}_r$ if $f(0) = 0$, and if $\mathcal{L}_2(x) \leqslant f(x) \leqslant \mathcal{L}_1(x)$ and $f'(x) \geqslant 1$ for all $x \in [0, r]$. Define $F_r\colon \mathcal{H}_r \to C^1([0, r])$ again by $(F_r(f))(x) = x + (f^{[-1]}(x))^2$. This is consistent with F_r's prior definition on functions in their common domain. We consider the derivative of $F_r(f)$:

$$(F_r(f))'(x) = 1 + \frac{2f^{[-1]}(x)}{f'(f^{[-1]}(x))}.$$

Then F_r maps $\mathcal{H}_r$ continuously (with the C^1 topology) into itself, since $\mathcal{L}_2(x) \leqslant f(x) \leqslant \mathcal{L}_1(x)$ implies $(F_r(\mathcal{L}_1))(x) = \mathcal{L}_2(x) \leqslant (F_r(f))(x) \leqslant \mathcal{L}_3(x) = (F_r(\mathcal{L}_2))(x) \leqslant \mathcal{L}_1(x)$, and $(F_r(f))'(x)$ is clearly at least 1.

When $f \in \mathcal{H}_r$, $f'(f^{[-1]}(x)) \geqslant 1$ and $f^{[-1]}(x)$ is increasing, so

$$1 \leqslant (F_r(f))'(x) = 1 + \frac{2f^{[-1]}(x)}{f'(f^{[-1]}(x))} \leqslant 1 + 2f^{[-1]}(r) \leqslant 1 + 2\mathcal{L}_2^{[-1]}(r).$$

If we define $\tilde{\mathcal{H}}_r = \{f \in \mathcal{H}_r : f'(x) \leqslant 1 + 2\mathcal{L}_2^{[-1]}(r) \text{ for } x \in [0, r]\}$, then $\tilde{\mathcal{H}}_r$ is a closed, bounded, convex set and $F_r(\tilde{\mathcal{H}}_r) \subset \tilde{\mathcal{H}}_r$.

We apply Darbo's Theorem to F_r acting on $\tilde{\mathcal{H}}_r$ with Ω_1 as the generalized measure of noncompactness. Let A be a bounded subset of $\tilde{\mathcal{H}}_r$. $\Omega_1(A)$ is the limit as $\delta \to 0^+$ of $\omega_\delta(A^{(1)})$, where $A^{(1)}$ is the set of derivatives of functions in A. Given $\delta > 0$, we estimate $|(F_r(f))'(s) - (F_r(f))'(t)|$ in terms of $|f'(s) - f'(t)|$ when $f \in A$ and $s, t \in [0, r]$ with $|s - t| \leqslant \delta$.

$$\begin{aligned}
&|(F_r(f))'(s) - (F_r(f))'(t)| \\
&\quad = \left| \frac{2f^{[-1]}(s)\, f'(f^{[-1]}(t)) - 2f^{[-1]}(t)\, f'(f^{[-1]}(s))}{f'(f^{[-1]}(s))\, f'(f^{[-1]}(t))} \right| \\
&\quad \leqslant |2f^{[-1]}(s)\, f'(f^{[-1]}(t)) - 2f^{[-1]}(t)\, f'(f^{[-1]}(s))| \\
&\quad \leqslant (2\,|f^{[-1]}(s) - f^{[-1]}(t)|)\, f'(f^{[-1]}(t)) \\
&\qquad + 2f^{[-1]}(t)(|f'(f^{[-1]}(t)) - f'(f^{[-1]}(s))|).
\end{aligned}$$

We examine these terms separately. $|f^{[-1]}(s)-f^{[-1]}(t)| \leqslant |s-t|$ since $0<(f^{[-1]})'(x) \leqslant 1$ for $x \in [0, r]$. Also $1 \leqslant f'(x) \leqslant 1+2\mathscr{L}_2^{[-1]}(r)$ for all $x \in [0, r]$, and therefore

$$(2\,|f^{[-1]}(s)-f^{[-1]}(t)|)\, f'(f^{[-1]}(t)) \leqslant 2\,|s-t|\,(1+2\mathscr{L}_2^{[-1]}(r)).$$

For the other term, observe that $f^{[-1]}(t) \leqslant \mathscr{L}_2^{[-1]}(r)$ and that both $f^{[-1]}(s)$ and $f^{[-1]}(t)$ are in $[0, r]$ and $|f^{[-1]}(s)-f^{[-1]}(t)| \leqslant \delta$ so

$$2f^{[-1]}(t)(|f'(f^{[-1]}(t))-f'(f^{[-1]}(s))|) \leqslant 2\mathscr{L}_2^{[-1]}(r)\, \omega_\delta(\{f' \colon f \in A\}).$$

We have thus shown

$$\omega_\delta(\{(F_r(f))' \colon f \in A\}) \leqslant 2\delta(1+2\mathscr{L}_2^{[-1]}(r))+2\mathscr{L}_2^{[-1]}(r)\, \omega_\delta(\{f' \colon f \in A\}).$$

Then taking limits as $\delta \to 0^+$ we see that $\Omega_1(F_r(A)) \leqslant 2\mathscr{L}_2^{[-1]}(r)\, \Omega_1(A)$.

Since $2\mathscr{L}_2^{[-1]}(r)<1$, Darbo's Theorem (Theorem 10.2) implies that F_r has a fixed point in $\widetilde{\mathscr{H}}_r$. But F_r has a unique fixed point in $G_r \supset \widetilde{\mathscr{H}}_r$, and therefore this fixed point must be C^1 in $[0, r]$. Now use the reasoning of the remark after Proposition 10.1 to "transmit" differentiability to all of $[0, \infty)$: since L is C^1 in $[0, r]$, $L^{[-1]}$ is C^1 in $[0, L(r)]$, and $(*)$ then shows that L is C^1 in $[0, L(r)]$. By induction we see that L is C^1 in $[0, L^{[k]}(r)]$ for any positive integer k, and since $\lim_{k \to \infty} L^{[k]}(r)=\infty$ we are done. ▮

The proof that L is C^m for any positive integer m follows the outline established above. Again we control the variation of the highest derivative with two terms. One is an "error term" depending on $\delta>0$, which vanishes as $\delta \to 0^+$. The other term satisfies the hypotheses of Darbo's Theorem. The algebra is more elaborate. We have tried to isolate that aspect in the following lemma which gives a formula for the mth derivative of $f^{[-1]}$.

Lemma 10.3. *Suppose $m \geqslant 2$, $f \in C^m([a, b])$, and $f'(x)>0$ for $x \in [a, b]$. Then there is a polynomial $\mathscr{P}_m \in \mathbb{Z}[w_1, w_2, ..., w_{m-1}]$ independent of a, b, and f, so that for $x \in [f(a), f(b)]$,*

$$(f^{[-1]})^{(m)}(x)=\mathscr{P}_m\left(\frac{1}{f'(u)}, f^{(2)}(u), ..., f^{(m-1)}(u)\right)-\left(\frac{1}{f'(u)}\right)^{m+1} f^{(m)}(u),$$

where $u=f^{[-1]}(x)$.

Proof. Since $(f^{[-1]})'(x)=1/f'(u)$ we may write $(f^{[-1]})''(x)=(-1/(f'(u))^2)\, f''(u)(1/f'(u))=(-1/(f'(u))^3)\, f''(u)$. So the result is verified for $m=2$ with $\mathscr{P}_2$ equal to 0.

Now assume for some $k \geqslant 2$ that $(f^{[-1]})^{(k)}$ has the desired form

$$\mathscr{P}_k\left(\frac{1}{f'(u)}, f^{(2)}(u), ..., f^{(k-1)}(u)\right) - \left(\frac{1}{f'(u)}\right)^{k+1} f^{(k)}(u),$$

and differentiate. The chain rule and the earlier formula for $(f^{[-1]})''$ produce

$$\sum_{j=1}^{k-1} D_j \mathscr{P}_k(w_1, w_2, ..., w_{k-1})\, w_j{}' + \frac{(k+1)}{(f'(u))^{k+3}} f''(u)\, f^{(k)}(u)$$
$$-\left(\frac{1}{f'(u)}\right)^{k+1} f^{(k+1)}(u)\, \frac{1}{f'(u)},$$

where D_j is the jth partial derivative of the polynomial $\mathscr{P}_k$. Here $w_1{}' = (-1/(f'(u))^3)\, f''(u) = -w_1^3 w_2$, and $w_j' = (f^{(j)}(u))' = f^{(j+1)}(u)\, u' = w_{j+1} w_1$ if $2 \leqslant j \leqslant k-1$. The polynomial

$$\begin{aligned}\mathscr{P}_{k+1}(w_1, w_2, ..., w_k) = {} & -D_1 \mathscr{P}_k(w_1, w_2, ..., w_{k-1})\, w_1^3 w_2 \\ & + \left(\sum_{j=2}^{k-1} D_j \mathscr{P}_k(w_1, w_2, ..., w_{k-1})\, w_{j+1} w_1\right) \\ & + (k+1)\, w_1^{k+3} w_2 w_k\end{aligned}$$

together with the preceding computation completes the induction proof. ∎

THEOREM 10.4. *Suppose L is the unique strictly increasing solution to $(*)$. Then $L \in C^\infty([0, \infty))$.*

Proof. Let m be an integer greater than 1. We prove that $L \in C^\infty([0, \infty))$. Select $r > 0$ which is less than $(3-\sqrt{3})/2$. We always consider $C^k([0, r])$ as a subset of $C^j([0, r])$ for $k \geqslant j \geqslant 0$. Let $\widetilde{\mathscr{H}}_r$ be defined as in the proof of Theorem 10.3. Then if $f \in \widetilde{\mathscr{H}}_r$, $f'(x) \geqslant 1$ for all $x \in [0, r]$, and $\mathscr{L}_2 \leqslant f \leqslant \mathscr{L}_1$. $\widetilde{\mathscr{H}}_r$ is closed, bounded, and convex in $C^1([0, r])$ and $\widetilde{\mathscr{H}}_r \subset C^1([0, r]) \cap G_r$. Also $F_r(\widetilde{\mathscr{H}}_r) \subset \widetilde{\mathscr{H}}_r$.

We define $G_{r,1}$ to be $\widetilde{\mathscr{H}}_r$. Assume that we have found for each j with $1 \leqslant j < m$, a subset $G_{r,j}$ of $C^j([0, r])$ which is closed, bounded, and convex in $C^j([0, r])$, that $F_r(G_{r,j}) \subset G_{r,j}$, and $G_{r,j+1} \subset G_{r,j}$ for $j < m-1$. We will find a closed, bounded, convex subset $G_{r,m}$ of $C^m([0, r])$ so that

(1) $G_{r,m} \subset G_{r,m-1}$.

(2) $F_r(G_{r,m}) \subset G_{r,m}$.

(3) $\Omega_m(F_r(A)) \leqslant 2\mathscr{L}_2^{[-1]}(r)\, \Omega_m(A)$ for all $A \subset G_{r,m}$.

In fact, $G_{r,m}$ will be $G_{r,m-1} \cap \{f \in C^m([0,r]) : \sup_{0 \leqslant x \leqslant r} |f^{(m)}(x)| \leqslant K_m\}$ for some suitable constant K_m. If $G_{r,m}$ is such a set, then it is surely closed, bounded, and convex in $C^m([0,r])$ and $G_{r,m} \subset G_{r,m-1}$. We now describe how to select K_m so that the other specifications listed are fulfilled.

If $f \in G_{r,m}$, then Leibniz's formula states that

$$(F_r(f))^{(m)}(x) = \sum_{j=0}^{m} \binom{m}{j} (f^{[-1]})^{(j)}(x) (f^{[-1]})^{(m-j)}(x)$$

for $x \in [0,r]$. The "x" term in $F_r(f)$ vanishes because we are taking at least two derivatives. We apply Lemma 10.3 to analyze this sum. We specially note the terms where $(f^{[-1]})^{(m)}(x)$ appears: there are two, when $j=0$ and $j=m$. Each such term is multiplied by $u=(f^{[-1]})^{(0)}(x)$. We can collect terms and declare that there is a polynomial $\mathscr{Q}_m \in \mathbb{Z}[w_1, w_2, ..., w_{m-1}]$ not depending on f so that

$$(F_r(f))^{(m)}(x) = \frac{-2u}{(f'(u))^{m+1}} f^{(m)}(u) + \mathscr{Q}_m\left(\frac{1}{f'(u)}, f^{(2)}(u), ..., f^{(m-1)}(u)\right). \tag{$**$}$$

By assumption, if $f \in G_{r,m-1}$, there is a constant D_m so that $\sup\{|f^{(j)}(x)| : 0 \leqslant j \leqslant m-1 \text{ and } x \in [0,r]\}$ is bounded by D_m. Thus there must be a constant E_m so that

$$\sup\left\{\left|Q_m\left(\frac{1}{f'(u)}, f^{(2)}(u), ..., f^{(m-1)}(u)\right)\right| : \right.$$
$$\left. f \in G_{r,m-1},\ u = f^{[-1]}(x), \text{ and } x \in [0,r]\right\}$$

is bounded by E_m.

If $f \in G_{r,m-1}$, then since $G_{r,m-1} \subset \tilde{\mathscr{H}}_r$, we know (see the proof of Theorem 10.3) $2u = 2f^{[-1]}(x) \leqslant 2\mathscr{L}_2^{[-1]}(r) < 1$. Also $f'(u) \geqslant 1$ always. Therefore

$$|(F_r(f))^{(m)}(x)| \leqslant 2\mathscr{L}_2^{[-1]}(r)\, |f^{(m)}(f^{[-1]}(x))| + E_m.$$

Select K_m so large that

$$2\mathscr{L}_2^{[-1]}(r)\, K_m + E_m \leqslant K_m$$

and define $G_{r,m}$ as written earlier. F_r now maps $G_{r.m}$ continuously into $G_{r,m}$.

We will bound $\Omega_m(F_r(A))$ by a multiple of $\Omega_m(A)$ for $A \subset G_{r,m}$, as described by (3). Let $(F_r(A))^{(m)}$ denote the set of mth derivatives of functions in $F_r(A)$. Equation $(**)$ allows us to write

$$(F_r(A))^{(m)} \subset A_1 + A_2,$$

where

$$A_1 = \left\{ \frac{-2u}{(f'(u))^{m+1}} f^{(m)}(u) \colon f \in A \text{ and } u = f^{[-1]}(x) \right\}$$

and

$$A_2 = \left\{ \mathcal{Q}_m\left(\frac{1}{f'(u)}, f^{(2)}(u), ..., f^{(m-1)}(u) \right) \colon f \in A \text{ and } u = f^{[-1]}(x) \right\}.$$

Of course, $\Omega_m(F_r(A)) = \omega((F_r(A))^{(m)})$, and by property (e) above, this is bounded by $\omega(A_1) + \omega(A_2)$. But A_2 is a bounded, equicontinuous subset of $C([0, r])$ (functions in A_2 have their derivatives are bounded because of the restriction given by K_m), and so $\overline{A_2}$ is compact, and $\omega(A_2) = 0$.

Therefore $\omega((F_r(A))^{(m)}) \leqslant \omega(A_1)$. Given $\delta > 0$, we now estimate $\omega_\delta(A_1)$ as in the proof of the previous theorem. If $s, t \in [0, r]$ with $|s - t| \leqslant \delta$, let $u = f^{[-1]}(s)$ and $v = f^{[-1]}(t)$. Consider

$$\left| \frac{2uf^{(m)}(u)}{(f'(u))^{m+1}} - \frac{2vf^{(m)}(v)}{(f'(v))^{m+1}} \right|$$

which we must estimate.

The function which takes the pair $(s, t) \in [0, r] \times [0, r]$ to

$$W(u, v) = \left| \frac{2u}{(f'(u))^{m+1}} - \frac{2v}{(f'(v))^{m+1}} \right|$$

vanishes on the diagonal and is smooth on its compact domain. Thus given $\delta > 0$, there is $M > 0$ so that $|W(u, v)| \leqslant M\delta$ when $|s - t| < \delta$. Then

$$\begin{aligned} &\left| \frac{2uf^{(m)}(u)}{(f'(u))^{m+1}} - \frac{2vf^{(m)}(v)}{(f'(v))^{m+1}} \right| \\ &\qquad \leqslant W(u, v)\, |f^{(m)}(u)| + \frac{2v}{(f'(v))^{m+1}}\, |f^{(m)}(u) - f^{(m)}(v)| \\ &\qquad \leqslant (M\delta)\, K_m + 2v\, |f^{(m)}(u) - f^{(m)}(v)| \leqslant K_M M\delta + 2\mathcal{L}_2^{[-1]}(r)\, \omega_\delta(A^{(m)}), \end{aligned}$$

where $A^{(m)}$ is the set of mth derivatives of functions in A. The estimation $|f^{(m)}(u) - f^{(m)}(v)| \leqslant \omega_\delta(A)$ occurs because $|u - v| = |f^{[-1]}(s) - f^{[-1]}(t)| \leqslant |s - t| \leqslant \delta$ since $0 < (f^{[-1]})'(x) \leqslant 1$ for $x \in [0, r]$. Of course, $f'(v) \geqslant 1$

which allows the omission of the powers of $f'(v)$ in the denominator to help the overestimate. Let $\delta \to 0^+$ to get $\omega(A_1) \leqslant 2\mathscr{L}_{\frac{1}{2}}^{[-1]}(r)\,\omega(A^{(m)})$, which implies (3).

Darbo's Theorem applies as before. F_r has a fixed point in $G_{r,m}$ which must be L restricted to $[0, r]$ by uniqueness of F_r on $\widetilde{\mathscr{H}}_r$. The functional equation can be used to show that L is C^m on all of $[0, \infty)$. Since $m > 1$ was arbitrary, L must be C^∞. ▮

11. HOMOCLINIC DOUBLY INFINITE COMPLEX SEQUENCES

We show below that there is a set of complex initial conditions for the QF recurrence with non-empty interior in $\mathbb{C}^2$ so that the resulting sequences always have limit 0. If both y_0 and y_1 are 0, $x_0 = -q$ and $x_1 = -qp^{-1}$ (so γ below is p^{-1}), a statement similar to Theorem 2.2 is recovered. The proof below is also similar in outline to that theorem's, but the details are more complicated.

THEOREM 11.1. *Suppose $\beta > 1$ and $\gamma > 1$ are real numbers. Let $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$ be complex numbers such that*

(1) $x_0 < 0$, $x_1 < 0$, $y_0 \geqslant 0$, *and* $y_1 \geqslant 0$;

(2) $y_1/|x_1| \leqslant 1$, $y_0/|x_0| \leqslant 1$, *and* $y_1/|x_1| \leqslant 2(y_0/|x_0|)$;

(3) $|x_1| \leqslant \min(\gamma^{-2} - \gamma^{-3}, \frac{1}{2}(\beta^{-1} - \beta^{-2}))$;

(4) $\gamma x_1 \leqslant x_0$;

(5) $((\beta - 1)/2\beta)\, y_1 \geqslant |x_0|\, y_0$.

For $j \geqslant 2$ define $z_j = x_j + iy_j$ by $z_j = z_{j-1} + z_{j-2}^2$. Then

(6) $x_{j-1} \leqslant x_j < 0$ *and* $y_{j-1} \geqslant y_j \geqslant 0$ *for all* $j \geqslant 2$;

(7) $\gamma x_j \leqslant x_{j-1}$ *for all* $j \geqslant 1$ *and* $\beta y_j \geqslant y_{j-1}$ *for all* $j \geqslant 2$;

(8) $y_j/|x_j| \leqslant y_{j-1}/|x_{j-1}|$ *for all* $j \geqslant 2$;

(9) $\lim_{j \to \infty} z_j = 0$.

Proof. We claim that for all $j \geqslant 2$ we have

$$\text{(a)} \quad x_{j-1} \leqslant x_j \leqslant \gamma^{-1} x_{j-1} < 0;$$

$$\text{(b)} \quad y_{j-1} \geqslant y_j \geqslant \beta^{-1} y_{j-1} \geqslant 0;$$

$$\text{(c)} \quad \frac{y_{j-1}}{|x_{j-1}|} \geqslant \frac{y_j}{|x_j|}.$$

We first establish these inequalities for $j=2$ and then prove them generally by induction. Of course $x_{j+1}=x_j+x_{j-1}^2-y_{j-1}^2$ and $y_{j+1}=y_j+2x_{j-1}y_{j-1}$ for $j\geqslant 1$.

Equations (1) and (2) show that $y_0^2\leqslant x_0^2$. Since $x_2=x_1+x_0^2-y_0^2$, $x_2\geqslant x_1$. Now $x_2\leqslant x_1+x_0^2$ and equations (1) and (4) show that $x_2\leqslant x_1+\gamma^2x_1^2=x_1(1-\gamma^2\,|x_1|)$. This implies $x_2\leqslant\gamma^{-1}x_1$ if $\gamma^{-1}\leqslant 1-\gamma^2\,|x_1|$, which is exactly guaranteed by part of (3). So (a) holds for $j=2$.

Since $y_2=y_1+2x_0\,y_0$, then by (1), $y_2\leqslant y_1$. Also $y_2\geqslant\beta^{-1}y_1$ is true if $y_1+2x_0\,y_0\geqslant\beta^{-1}y_1$ which is exactly equation (5) since x_0 is negative and y_0 is non-negative, and (b) is true for $j=2$.

We relate $y_2/|x_2|$ to $\lambda=y_1/|x_1|$:

$$\frac{y_2}{x_2}=\frac{y_1+2x_0\,y_0}{-x_1-x_0^2+y_0^2}=\lambda\,\frac{\left(1-\dfrac{2\,|x_0|\,y_0}{\lambda\,|x_1|}\right)}{\left(1-\dfrac{(x_0^2-y_0^2)}{|x_1|}\right)}.$$

Therefore $y_2/|x_2|\leqslant y_1/|x_1|$ is true if

$$\left(1-\frac{2\,|x_0|\,y_0}{\lambda\,|x_1|}\right)\leqslant\left(1-\frac{x_0^2-y_0^2}{|x_1|}\right)$$

and that in turn is the same as $(x_0^2-y_0^2)\,\lambda\leqslant 2\,|x_0|\,y_0$. But in (2) we see that $y_1/|x_1|\leqslant 2(y_0/|x_0|)$ which can be rewritten $\lambda x_0^2\leqslant 2\,|x_0|\,y_0$, even stronger than we need. So (c) is also verified for $j=2$.

We now complete the inductive proof of (a), (b), and (c). We assume that these statements are true for all integers j where $2\leqslant j\leqslant k$, and we must prove them for $k+1$. We know from (c) that

$$1\geqslant\frac{y_1}{|x_1|}\geqslant\frac{y_2}{|x_2|}\geqslant\cdots\geqslant\frac{y_{k-1}}{|x_{k-1}|}$$

so $|x_{k-1}|^2\geqslant y_{k-1}^2$. Since $x_{k+1}=x_k+x_{k-1}^2-y_{k-1}^2$, $x_{k+1}\geqslant x_k$.

We will overestimate x_k. Consider the inequalities $x_{k+1}\leqslant x_k+x_{k-1}^2\leqslant x_k+\gamma^2x_k^2=x_k(1-\gamma^2\,|x_k|)$. x_{k+1} will be overestimated by $\gamma^{-1}x_k$ if $1-\gamma^2\,|x_k|\geqslant\gamma^{-1}$ (because x_k is negative). $1-\gamma^2\,|x_k|\geqslant\gamma^{-1}$ is true if $\gamma^{-2}-\gamma^{-3}\geqslant|x_k|$. But $x_1\leqslant x_k<0$ so $|x_1|\geqslant|x_k|$ and assumption (3) provides the link: $|x_1|\leqslant\gamma^{-2}-\gamma^{-3}$. (a) is therefore true for $j=k+1$.

The relations $y_{k+1}=y_k+2x_{k-1}\,y_{k-1}$, $x_{k-1}<0$, and $y_{k-1}\geqslant 0$ imply that $y_{k+1}\leqslant y_k$. $\beta y_{k+1}\geqslant y_k$ is true if $\beta(y_k+2x_{k-1}\,y_{k-1})\geqslant y_k$. This is the same as requiring that

$$(\beta-1)\,y_k\geqslant -2\beta x_{k-1}\,y_{k-1}=2\beta\,|x_{k-1}|\,y_{k-1}.$$

Our inductive hypothesis states that $\beta y_k \geqslant y_{k-1}$ so we are done bounding y_{k+1} if we know that $(\beta-1)\, y_k \geqslant 2\beta^2\, |x_{k-1}|\, y_k$. This last inequality is true if $\beta-1 \geqslant 2\beta^2\, |x_{k-1}|$. But we know $|x_{k-1}| \leqslant |x_1|$ and from (3) that $|x_1| \leqslant \frac{1}{2}(\beta^{-1}-\beta^{-2})$ so (b) is verified for $j=k+1$.

We know that x_{k+1} is negative and y_{k+1} is non-negative. Therefore to prove (c) for $j=k+1$ we must show that

$$\frac{y_{k+1}}{|x_{k+1}|} = \frac{y_k + 2x_{k-1}y_{k-1}}{-x_k - x_{k-1}^2 + y_{k-1}^2} \leqslant \tilde{\lambda} = \frac{y_k}{|x_k|}.$$

As before,

$$\frac{y_k + 2x_{k-1}\, y_{k-1}}{-x_k - x_{k-1}^2 + y_{k-1}^2} = \tilde{\lambda}\, \frac{\left(1 - \dfrac{2\, |x_{k-1}|\, y_{k-1}}{\tilde{\lambda}|x_k|}\right)}{\left(1 - \dfrac{(x_{k-1}^2 - y_{k-1}^2)}{|x_k|}\right)}.$$

So (c) is proved if

$$1 - \frac{2\, |x_{k-1}|\, y_{k-1}}{\tilde{\lambda}\, |x_k|} \leqslant 1 - \frac{(x_{k-1}^2 - y_{k-1}^2)}{|x_k|}$$

which is the same as $\tilde{\lambda}(x_{k-1}^2 - y_{k-1}^2) \leqslant 2\, |x_{k-1}|\, y_{k-1}$. It is certainly sufficient to verify that $\tilde{\lambda} x_{k-1}^2 \leqslant 2\, |x_{k-1}|\, y_{k-1}$ holds, an inequality equivalent to $y_k/|x_k| \leqslant 2(y_{k-1}/|x_{k-1}|)$. But (c) for $j=k$ asserts this is correct with 2 replaced by 1, so we have completed the inductive proof of (c).

We still must prove assertion (9). $\{x_k\}$ is an increasing sequence of negative reals, and $\{y_k\}$ is a decreasing sequence of non-negative reals. Therefore each sequence has a limit:

$$\lim_{k\to\infty} x_k = x \leqslant 0 \qquad \text{and} \qquad \lim_{k\to\infty} y_k = y \geqslant 0.$$

Then $z = \lim_{k\to\infty} z_k = x + iy$ and $\lim_{k\to\infty} z_{k+2} = z = \lim_{k\to\infty} z_{k+1} + z_k^2 = z + z^2$, so $z^2 = 0$ and $z = 0$. ∎

Since complex conjugation commutes with Ψ, if $(z_0, z_1) \in \mathbb{C}^2$ is chosen so that $(\overline{z_0}, \overline{z_1})$ satisfies the hypotheses of the theorem (statements (1) through (5)), then the sequence $\{\Psi^{[k]}(z_0, z_1)\}$ must also approach $(0, 0)$. In this case, for all $k \geqslant 2$, the y_k's are non-positive, $|y_{k-1}| \geqslant |y_k| \geqslant \beta^{-1}\, |y_{k-1}|$, and $|y_k|/|x_k| \leqslant |y_{k-1}|/|x_{k-1}|$.

Definition 11.1. A complex sequence $\{z_j\}_{j\in\mathbb{Z}}$ is *homoclinic* for the QF recurrence if some z_j is not 0, if $z_j^2 + z_{j+1} = z_{j+2}$ for all $j \in \mathbb{Z}$, and if $\lim_{j\to\infty} z_j = 0$ and $\lim_{j\to-\infty} z_j = 0$

Classically, one investigates homoclinic points of a homeomorphism $\Theta: X \to X$, where X is a metric space (see, e.g., p. 275 of [11]). Our map Ψ is not a homeomorphism or a local homeomorphism near $(0, 0)$ but our definition is a natural extension of the classical one. Homoclinic sequences exist for the QF recurrence. Recall that we have shown in Theorem 8.10 that for every $z \in \mathbb{C}_-$, there exists an *AIR* sequence $\{z_j\}_{j \in \mathbb{Z}}$ with $z_0 = z$.

THEOREM 11.2. *Suppose that* $3\pi/4 < \theta < \pi$ *and* $\beta > 1$ *and* $\gamma > 1$ *are real numbers. Then there is a positive number* $\rho(\theta, \beta, \gamma)$ *with the following properties*:

If $0 < \rho < \rho(\theta, \beta, \gamma)$ *and if* $\{z_j\}_{j \in \mathbb{Z}}$ *is an AIR sequence through* $z = \rho e^{i\theta}$, *then*

(1) $0 < \arg z_j \leqslant \arg z_{j+1} < \pi$ *for all* $j \in \mathbb{Z}$;

(2) *If* $x_j = \operatorname{Re} z_j$ *and* $y_j = \operatorname{Im} z_j$, *then* $\gamma x_j \leqslant x_{j-1} \leqslant x_j < 0$ *and* $0 \leqslant y_j \leqslant y_{j-1} \leqslant \beta y_j$ *for all* $j \geqslant 1$;

(3) $\lim_{j \to \infty} z_j = 0$ *and* $\lim_{j \to -\infty} z_j = 0$.

Proof. By Theorem 8.1 there exists an *AIR* sequence $\{z_j\}_{j \in Z}$ with $z_0 = z$. Proposition 8.1 asserts that $\lim_{j \to -\infty} z_j = 0$ and establishes (1) for $j \leqslant 0$. The other conclusions of the theorem will follow if we can show that the hypotheses of the preceding theorem are satisfied. Conclusion (8) of that theorem establishes statement (1) of this theorem when $j \geqslant 0$ for the values of θ under consideration (the quotients in (8) are just $|\tan(\arg z_j)|$). So we must verify conditions (1) through (5) of Theorem 11.1. Here we consider $\theta \in (3\pi/4, \pi)$, $\beta > 1$, and $\gamma > 1$ as constants, and show how to pick sufficiently small ρ's satisfying these conditions. Note that the variables x_0, y_0, x_1, and y_1 in the hypotheses of Theorem 11.1 are here called x_{-1}, y_{-1}, x_0, and y_0 respectively.

We have shown that there is $\kappa_2 > 0$ so that $|z_{j-1}| \leqslant \kappa_2 |z_j|$ for $j \leqslant 0$, and κ_2 depends only on θ. (For this see the proof of Lemma 8.4 and the discussion following the proof of Lemma 8.5.) Therefore $|z_{-1}| \leqslant \kappa_2 |z_0| \leqslant \kappa_2 \rho$ and $|z_{-2}| \leqslant \kappa_2 |z_{-1}| \leqslant \kappa_2^2 \rho$. Since $|z_0 - z_{-1}| = |z_{-2}^2|$, we conclude that $|z_0 - z_{-1}| \leqslant \kappa_2^4 \rho^2$. The real and imaginary parts of this expression follow

$$|x_0 - x_{-1}| \leqslant |z_0 - z_{-1}| \leqslant \kappa_2^4 \rho^2 \qquad \text{and} \qquad |y_0 - y_{-1}| \leqslant |z_0 - z_{-1}| \leqslant \kappa_2^4 \rho^2$$

and, writing $x_0 = \rho \cos \theta$ and $y_0 = \rho \sin \theta$, we have

$$\rho \cos \theta - \kappa_2^4 \rho^2 \leqslant x_{-1} \leqslant \rho \cos \theta + \kappa_2^4 \rho^2 \qquad \text{and}$$
$$\rho \sin \theta - \kappa_2^4 \rho^2 \leqslant y_{-1} \leqslant \rho \sin \theta + \kappa_2^4 \rho^2.$$

Recall that $\theta \in (3\pi/4, \pi)$ so $\sin \theta > 0$. Suppose now $0 < \rho < \sin \theta / \kappa_2^4$. Then $\rho \sin \theta - \kappa_2^4 \rho^2 > 0$ and y_{-1} must be positive. Also for $\theta \in (3\pi/4, \pi)$, $|\cos \theta| >$

$\sin\theta$, so that $\kappa_2^4\rho^2<\rho\,|\cos\theta|$, forcing x_{-1} to be negative. Condition (1) is verified (that $x_0<0$ and $y_0>0$ follows from θ's restriction).

We estimate

$$\frac{y_{-1}}{|x_{-1}|}\leqslant\frac{\rho\sin\theta+\kappa_2^4\rho^2}{|\rho\cos\theta+\kappa_2^4\rho^2|}=\frac{\sin\theta+\kappa_2^4\rho}{|\cos\theta|-\kappa_2^4\rho}.$$

This is bounded by 1 for sufficiently small ρ since for θ fixed in $(3\pi/4,\pi)$, $0<|\tan\theta|<1$. The ratio $y_0/|x_0|$ is $|\tan\theta|$, so we have checked another part of (2). Now the corresponding underestimate is

$$2\,\frac{y_{-1}}{|x_{-1}|}\geqslant 2\,\frac{\rho\sin\theta-\kappa_2^4\rho^2}{|\rho\cos\theta-\kappa_2^4\rho^2|}=2\,\frac{\sin\theta-\kappa_2^4\rho}{|\cos\theta|+\kappa_2^4\rho}.$$

Again, for ρ small (θ is fixed!), this is close to $2\,|\tan\theta|$, which is certainly larger than $y_0/|x_0|$. All of (2) has been verified for small enough ρ.

Since $|x_0|=\rho\,|\cos\theta|\neq 0$, condition (3) can be satisfied by choosing ρ so that

$$\rho<|\cos\theta|^{-1}\min(\gamma^{-2}-\gamma^{-3},\tfrac{1}{2}(\beta^{-1}-\beta^{-2})).$$

We have now chosen ρ so that both x_0 and x_{-1} are negative. Condition (4) then becomes $\gamma\,|x_0|\geqslant|x_{-1}|$. But $|x_{-1}|\leqslant|x_0|+|x_0-x_{-1}|\leqslant\rho\,|\cos\theta|+\kappa_2^4\rho^2$ so we need to verify

$$\gamma\rho\,|\cos\theta|\geqslant\rho\,|\cos\theta|+\kappa_2^4\rho^2.$$

This holds if $0<\rho<\kappa_2^{-4}(\gamma-1)\,|\cos\theta|$ and therefore (4) is verified for sufficiently small ρ.

We know that $|x_{-1}|\leqslant\rho\,|\cos\theta|+\kappa_2^4\rho^2$ and $y_{-1}\leqslant\rho\sin\theta+\kappa_2^4\rho^2$. Thus condition (5) follows if the estimate

$$\left(\frac{\beta-1}{2\beta}\right)\rho\sin\theta\geqslant(\rho\,|\cos\theta|+\kappa_2^4\rho^2)(\rho\sin\theta+\kappa_2^4\rho^2)$$

holds. Again, this is clearly true for small positive ρ when θ and β are fixed. ∎

A specific complex number satisfying the conditions of the previous theorem is $z=0.001e^{2.5i}\approx -0.000\ 80114\ 36155+0.000\ 59847\ 21441i$, with $\beta=\gamma=2$ and $\kappa_2\approx 2.338$.

Remark. The most familiar recurrences are *linear*. Each of these is defined by a linear map, $T\colon\mathbb{C}^n\to\mathbb{C}$, which is applied recursively to an initial condition $Z=(z_0^*,z_1^*,\ldots,z_{n-1}^*)\in\mathbb{C}^n$. If $T(1,0,\ldots,0)\neq 0$, then there is always a unique doubly infinite sequence $\{z_n\}_{n\in\mathbb{Z}}$ defined by requiring that

$z_N = T(z_{N-n}, z_{N-n+1}, ..., z_{N-1})$ be true for all $N \in \mathbb{Z}$ and $z_j = z_j^*$ for $0 \leqslant j \leqslant n-1$. For example, the standard Fibonacci sequence is defined by $T(w, z) = (z, w+z)$ with initial condition $(0, 1)$. Linear recurrences cannot have homoclinic sequences so the qualitative behavior of the QF recurrence is quite different. The map $\Psi(w, z) = (z, z + w^2)$ which generates the QF recurrence has a fixed point at $(0, 0)$. Its linearization at $(0, 0)$ is $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with eigenvalues 0 and 1. The existence of homoclinic sequences demonstrates a striking consequence of nonlinearity.

Suppose $\zeta_j = (z_j, z_{j+1}) \in \mathbb{C}^2$ where $\{z_j\}$ is one of the sequences created in Theorem 11.2. Then $\Psi(\zeta_j) = \zeta_{j+1}$ for all $j \in \mathbb{Z}$, none of the ζ_j's are zero, $\lim_{j \to \infty} \zeta_j = 0 \in \mathbb{C}^2$, and $\lim_{j \to -\infty} \zeta_j = 0 \in \mathbb{C}^2$. Given $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$, an appropriate one-variable analogue of our homoclinic sequences might be a sequence $\{x_j\}_{j \in \mathbb{Z}}$ with $f(x_j) = x_{j+1}$ for all $j \in \mathbb{Z}$, and with $\lim_{j \to \infty} x_j = 0$ and $\lim_{j \to -\infty} x_j = 0$. Such sequences with all x_j non-zero can exist, even for polynomials: if $f(x) = x - x^2 - 0.29x^3$, choosing x_0 in a suitable neighborhood of the negative critical number of f (approximately -2.72124) guarantees the behavior of the sequence. Figure 8 illustrates the evolution of one of these sequences.

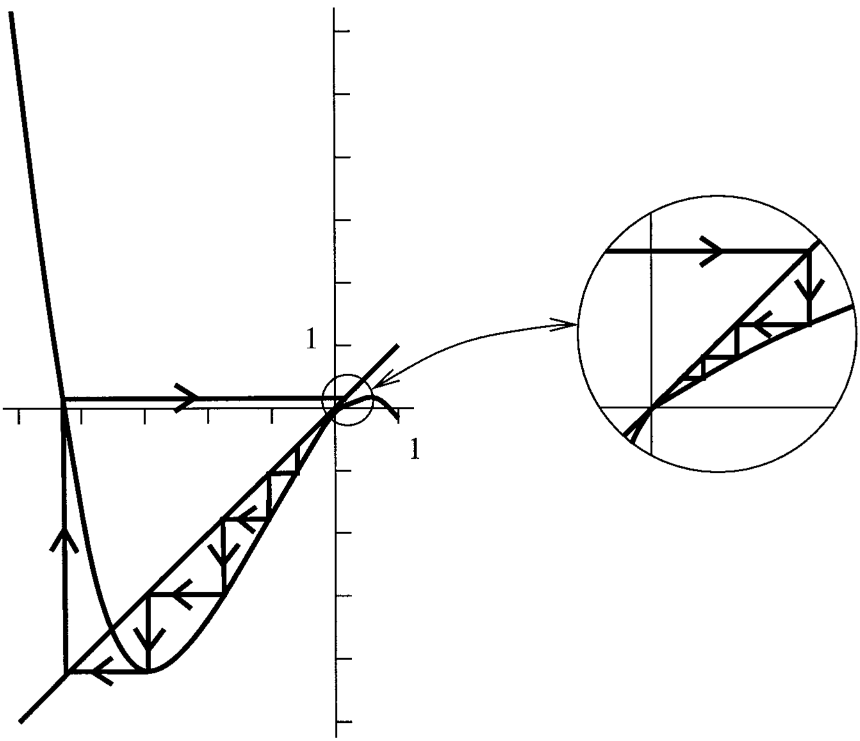

FIG. 8. x and $x - x^2 - 0.29x^3$.

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